In general, if we have a ~~non~~ fully nonlinear ~~ellip~~ eqn:

$$F(D^2u) = f. \qquad F(r): S^{n\times n} \to \mathbb{R}$$

F is elliptic if $\frac{\partial F}{\partial r_{ij}} > 0$.

uniformly elliptic if $\exists\, \lambda > 0, \Lambda > 0$ s.t.

$$\lambda I \le \frac{\partial F}{\partial r_{ij}} \le \Lambda I$$

Now, if $F(r) = \det r$, then

$\frac{\partial F}{\partial r_{ij}} = (\mathrm{cof}\, r)_{ij}$ Hence MA

is elliptic iff $D^2u > 0$ i.e. u convex

BUT MA is not uniformly elliptic

(unless you have a bound for D^2u)

Source of MA eqn & Application:

① ~~prescribing graphs of Gauss curvature~~

~~$u: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ (which x?)~~

Monge-Ampere equation (elliptic)

real version: ~~det~~ $u: \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

$\det D^2 u = f \rightsquigarrow$ given function $f(x)$. $f(x, u, \nabla u)$

A natural generalization of Poisson equation:

$$\Delta u = f.$$

Connection between MA eqn & Poisson's equation:

$\lambda_1 \cdots \lambda_n$ be the eigenvalues of $D^2 u$.

then Poisson: $\sum_{i=1}^{n} \lambda_i = f.$

$$\text{MA} : \prod_{i=1}^{n} \lambda_i = f.$$

Eqn in between: m- Hessian eqn. $1 \le m \le n$.

$$\sigma_m(\lambda) = f.$$

σ_m = m-th symmetric polynomial.

Q: When is MA eqn elliptic?

~~Ans~~ (Non-elliptic eqn is very difficult...)

Good news: The later problem always have a solution (possibly ~~non-unique~~ direct min)

~~Such a~~ soln $\gamma = (id \times T)_{\#}\mu$ iff ~~$\forall (x, y) \in supp(\gamma)$~~

for μ-a.e. x. if $(x, y_1), (x, y_2) \in supp(\gamma)$

$y_1 = y_2$. i.e. γ doesn't "split mass of μ".

~~Examples~~ Example: $\nu = \delta_a$.

$\Pi(\mu, \nu) = \mu \times \delta_a$. $T(X) = \{a\}$

~~$\gamma(E) = \gamma(E_1 \times E_2$~~

Example: $X = \{x_1 \cdots x_n\}$ $Y = \{y_1 \cdots y_n\}$.

$$\mu = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}. \quad \nu = \frac{1}{n}\sum_{j=1}^{n} \delta_{y_j}$$

$$\Pi(\mu, \nu) = \{\gamma_{ij}\}. \quad \gamma_{ij} = \gamma(\{x_i\} \times \{y_j\})$$

$$\sum_{i=1}^{n} \gamma_{ij} = 1 \,\forall j \quad \sum_{j=1}^{n} \gamma_{ij} = 1 \quad \forall i$$

$$\int c(x, y)\, d\gamma = \sum_{i, j} c(x_i, y_j)\gamma_{ij}.$$

① prescribing the Gauss curvature (positive) of a graph

$$K(x) = \frac{1}{(1+|Du|^2)^{\frac{n}{2}+1}} \det D^2 u$$

$\Omega \subseteq \mathbb{R}^n$ a convex domain. $u|_{\partial\Omega} = \varphi$ & Gauss curvature $= K(x)$.

② optimal transport (in ~~the sim~~ certain special setting)

Th. $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be two domain. $f > 0$ in Ω_1, $g > 0$ in Ω_2 with

$$\int_{\Omega_1} f\,dx = \int_{\Omega_2} g\,dx$$

Look for a Borel map $T: \Omega_1 \to \Omega_2$

~~st which minimizes $\int |x - T(x)| f(x) dx$~~

with $T_{\#}(f\,dx) = g\,dy$, which minimizes $\int_{\Omega_1} |x - T(x)| f(x)\,dx$

Thm. (1) The optimal map is unique, given by the gradient of a convex funct

(2) $\det D^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}$ $\quad (*)$

(3) ~~$\exists$ a convex function of~~ s.t.
Let ψ be the Legendre transform of φ. then ~~$\nabla\psi$~~ $S = D\psi$ minimizes

$$\int_{\Omega_2} |y - S(y)|^2 g(y)\,dy$$ over all maps

$$S_{\#}(g\,dy) = f\,dx$$

So the regularity of optimal maps is equivalent to studying the regularity of $(*)$

~~Opti~~ General set-up of optimal transport problem

Formulation of the Problem (Monge)

(X, μ), (Y, ν) be two measure spaces, with
Σ_1 $\quad \Sigma_2$

$$\mu(X) = \nu(Y) = 1$$

Consider all measurable maps

$$T : X \to Y \text{ s.t. } T_\# \mu = \nu.$$

~~Wish~~ Let $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a measurable function.

Wish to minimize

$$\int_X c(x, T(x))\, d\mu(x).$$

$\forall g \in L^\infty(Y, d\nu)$

$$\int g(y)\, d\nu(y) = \int g(T(x))\, d\mu(x)$$

~~Very~~ In general very difficult to tell if there is such a map

(Kantorovitch formulation)

~~First~~ Idea: Relax the problem to allow for "transport plans" i.e ~~consider~~ we replace ~~optimal map~~ T with the "$T_\# \mu = \nu$" to "$\gamma \in \Pi(\mu, \nu)$" γ on $X \times Y$ s.t $(\mathrm{proj}_1)_\# \gamma = \mu$ $(\mathrm{proj}_2)_\# \gamma = \nu$.

& minimizes $\int_{X \times Y} c(x, y)\, d\gamma(x, y)$