Recall thm from last time:

$\mu, \nu \in P(\mathbb{R}^n)$. $\int |x|^2 d\mu, d\nu < \infty$. $c(x,y) = \frac{1}{2}|x-y|^2$.

(1). $\pi_0 \in \Pi(\mu, \nu)$ is a minimizer to Kantorovitch iff $y \in \partial\varphi(x)$ for π-a.e (x, y).

(2) If $\mu \ll dm$ then $\exists!$ optimal π_0. s.t $\pi_0 = (id \times \nabla\varphi)_\# \mu$. for some convex φ.

(3) If $\nu \ll dm$. then the two optimal maps are inverse to each other (in the sense of a.e)

$\mu \xrightarrow{\nabla\varphi} \nu$ then $\nu \xrightarrow{\nabla\varphi^*} \mu$

φ^* = Legendre transform of φ

i.e $\varphi^*(y) = \sup_x (x \cdot y - \varphi(x))$.

Brenier's polar factorization thm.

$\Omega \subseteq \mathbb{R}^n$ bdd subset, $m(\Omega)>0$. $h: \Omega \to \mathbb{R}^n \in L^2$. s.t.

$\forall N \subseteq \mathbb{R}^n$, $m(N)$. $m(h^{-1}(N))=0$. (non degenerate condition).

Then $\exists$ convex function. s.t $h = \nabla \psi \circ S$, where ψ is convex. &

$S: \Omega \to \Omega$ measure preserving

$m(S^{-1}(A)) = m(A)$.

Moreover, this decomposition is unique. Also

$$\|S-h\|_{L^2} = \inf_{\substack{\sigma: \Omega \to \Omega \\ \text{measure preserving}}} \|\sigma - h\|_{L^2}$$

proof (may be done later). follows from applying the previous thm to: $\mathcal{L}|_\Omega$, $h_\#(\mathcal{L}|_\Omega)$

non-degenerate condition $\Longrightarrow h_\#(\mathcal{L}|_\Omega) \ll \mathcal{L}$.

Why called polar factorization thm?

Recall the following result from linear algebra:

Let $M \in M_{n\times n}(\mathbb{R}^n)$, then $\exists\, O \in O_n(\mathbb{R})$ & $S \in S_n^+(\mathbb{R})$. s.t $M = S \cdot O$.

Moreover, such O can be characterized as:

$$MO^{-1} \in S_n^+(\mathbb{R}) \iff \|M - O\|_{HS}^2 = \inf_{\tilde{O} \in O_n(\mathbb{R})} \|M - \tilde{O}\|_{HS}^2.$$

HS = Hilbert Schmidt norm. $\|M\|_{HS}^2 = \left(\sum_{i,j} M_{ij}^2\right)^{\frac{1}{2}}$.

If M is invertible, then S, & O is unique.

Relation between this p.f & Brenier's p.f:

$\Omega = B_1(0)$. $h : \Omega \to \mathbb{R}^n$. $hx = M \cdot x$. h non-degenerate $\iff$ M invertible.

then $h = \underline{\nabla\psi} \circ S$. (Since h is linear, can pretend $\nabla\psi$ & S is linear $|\det S| = 1$).

S linear. $\det S = 1$. $S(B_1(0)) \subseteq B_1(0) \Rightarrow S \in O_n(\mathbb{R})$

Relation to Helmholtz decomposition:

$\Omega \subseteq \mathbb{R}^n$ smooth domain, $\vec{w}$ a smooth v.f. then $\vec{w} = \vec{v} + \nabla p$.

$\mathrm{div}\, \vec{v} = 0$, $\vec{v} \cdot \vec{n} = 0$ on $\partial\Omega$.

<u>pf</u>: Let h_t be the flow generated from $\vec{w}$. $\frac{dh_t}{dt} = \vec{w}(h_t)$.

Then apply Brenier's thm. $h_t = \nabla \varphi_t \circ S_t$. φ_t is convex

$S_t : \Omega \to \Omega$ measure preserving map.

Assume everything is smooth.

$$\vec{w} = \frac{dh_t}{dt}\Big|_{t=0} = \frac{d}{dt}\left(\nabla\varphi_t \circ S_t\right)\Big|_{t=0} = \nabla \frac{d\varphi_t}{dt} \circ S_t\Big|_{t=0} + D^2\varphi_t \cdot \frac{dS_t}{dt}\Big|_{t=0} = \nabla \frac{d\varphi_t}{dt}\Big|_{t=0} + \frac{dS_t}{dt}\Big|_{t=0}$$

Since at $t=0$, $D^2\varphi_t = I$.
$S_t = id$.

proof of Brenier's factorization thm:

non-degenerate condition $\Rightarrow h_\#(\mathcal{L}|_\Omega) \ll \mathcal{L}|_\Omega$.

Pf: WLOG, assume $m(\Omega)=1$.

(check: $m(N)=0$. $h_\#(\mathcal{L}|_\Omega)(N) = m(h^{-1}(N))=0$)

Then $\exists$ convex function φ. & its legendre transform φ_*. s.t.

$$\nabla\varphi_\#(\mathcal{L}|_\Omega) = h_\#(\mathcal{L}|_\Omega). \quad \& \quad \mathcal{L}|_\Omega = (\nabla\varphi_*)_\# h_\#(\mathcal{L}|_\Omega).$$

So $S = \nabla\varphi_* \circ h$. then $S: \Omega \to \Omega$ measure preserving map.

& $h = \nabla\varphi \circ S$. (So we get the decomposition...)

Why $\|h-s\|_{L^2} = \inf_{\sigma\in S(\Omega)} \|h-\sigma\|_{L^2}$?

First, you can write down the Kantorovitch problem between $\mathcal{L}|_\Omega$, $h_\#(\mathcal{L}|_\Omega)$.

$\inf\left\{\int |x-y|^2 d\gamma : \gamma \in \Pi(\mathcal{L}|_\Omega, h_\#(\mathcal{L}|_\Omega))\right\}$. Note $(\sigma\times h)_\# \mathcal{L}|_\Omega \in \Pi(\mathcal{L}|_\Omega, h_\#\mathcal{L}|_\Omega)$

Since $\nabla\varphi$ is the optimal map s.t. $\nabla\varphi_\#(\mathcal{L}|_\Omega) = h_\#(\mathcal{L}|_\Omega)$.

$$\int_\Omega |x-\nabla\varphi(x)|^2 dx \leq \int |x-y|^2 d((\sigma\times h)_\# \mathcal{L}_\Omega) = \int_\Omega |\sigma(x)-h(x)|^2 dx$$

$$\| $$

$$\int |x-y|^2 d((id\times\nabla\varphi)_\# \mathcal{L}_\Omega)$$

$$\| $$

$$\int |x-y|^2 d((\nabla\varphi_*\times id)_\# h_\# \mathcal{L}_\Omega)$$

$$\nabla\varphi_* \circ h = S$$

$$\| $$

$$\int |x-y|^2 d((S\times h)_\# \mathcal{L}_\Omega)$$

$$= \int_\Omega |S(x)-h(x)|^2 dx$$

Now we come to regularity question of optimal transport / MA eqn.

Special case: $\mu = f(x)dx$, $\nu = g(y)dy$. φ convex. $\nabla\varphi_\#(f dx) = g dy$.

Eqn for φ (assume it's convex). $\quad D\varphi_{\#}(f(x)dx) = g(y)dy$

Let $\zeta(y)$ be a test function.

$$\int \zeta(y) g(y) dy = \int \zeta(y) d(D\varphi)_{\#}(f(x)dx) = \int \zeta(D\varphi(x)) \underset{\sim}{f(x)} dx$$

$\| \; y = D\varphi(x)$

$$\int \zeta(D\varphi(x)) \underset{\sim\sim\sim}{g(D\varphi(x)) \det D^2\varphi(x)} dx$$

Since ζ is arbitrary. $\quad \underline{g(D\varphi(x)) \det D^2\varphi(x) = f(x).} \quad (*)$

This motivates a possible definition of weak soln to $(*)$.

we say a convex function φ solves $(*)$ weakly, if

$$D\varphi_{\#}(f(x)dx) = g(y)dy. \quad \text{(Brenier solution).}$$

Other notions of weak solution:

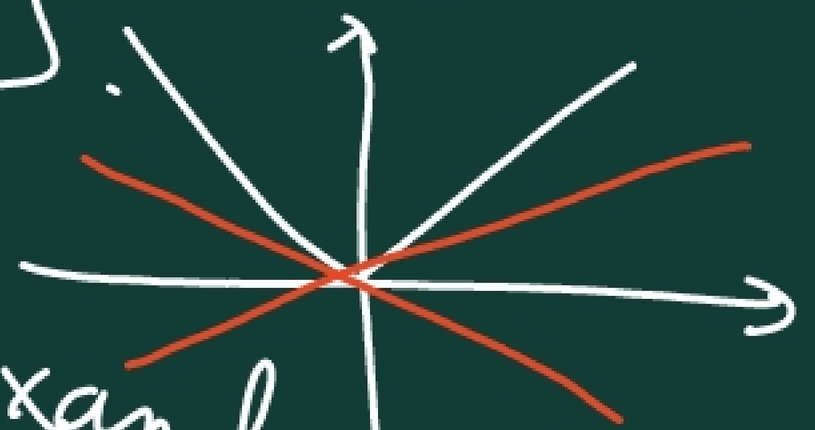
(1) (Alexandrov solution).

φ convex. we can define its Hessian measure by

$$\mu E = m(\partial\varphi(E)). \qquad \partial\varphi(E) = \bigcup_{x\in E} \partial\varphi(x).$$

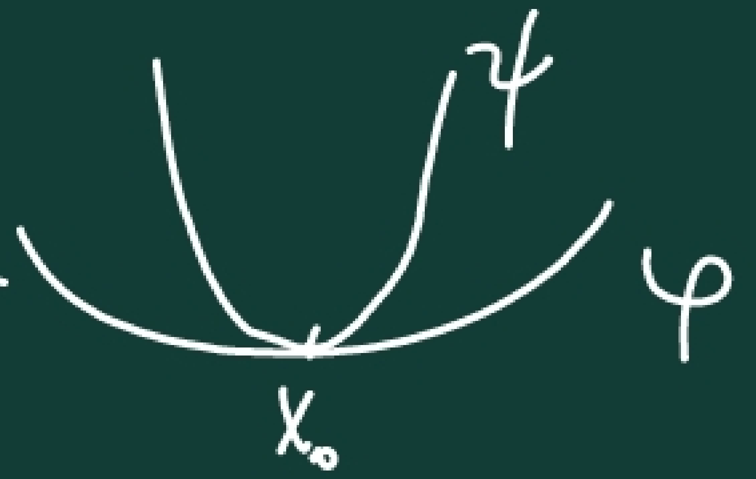
$$\partial\varphi(x) = \left\{ p\in\mathbb{R}^n : \varphi(y) \geq \varphi(x) + p\cdot(y-x) \quad \forall y\in\mathbb{R}^n \right\}.$$

e.g.: $\varphi(x) = |x|$. $\partial\varphi(\{0\}) = [-1, 1]$.



we say $\det D^2\varphi(x) = f(x, \varphi(x), \nabla\varphi(x))$ in the Alexandrov sense, if $\mu = f(x, \varphi, D\varphi)\,dx$

(2) (Viscosity solution).

(1). 

$\forall C^2$ convex ψ ($\psi(x_0) = \varphi(x_0)$, $\psi(x) \geq \varphi(x)$).

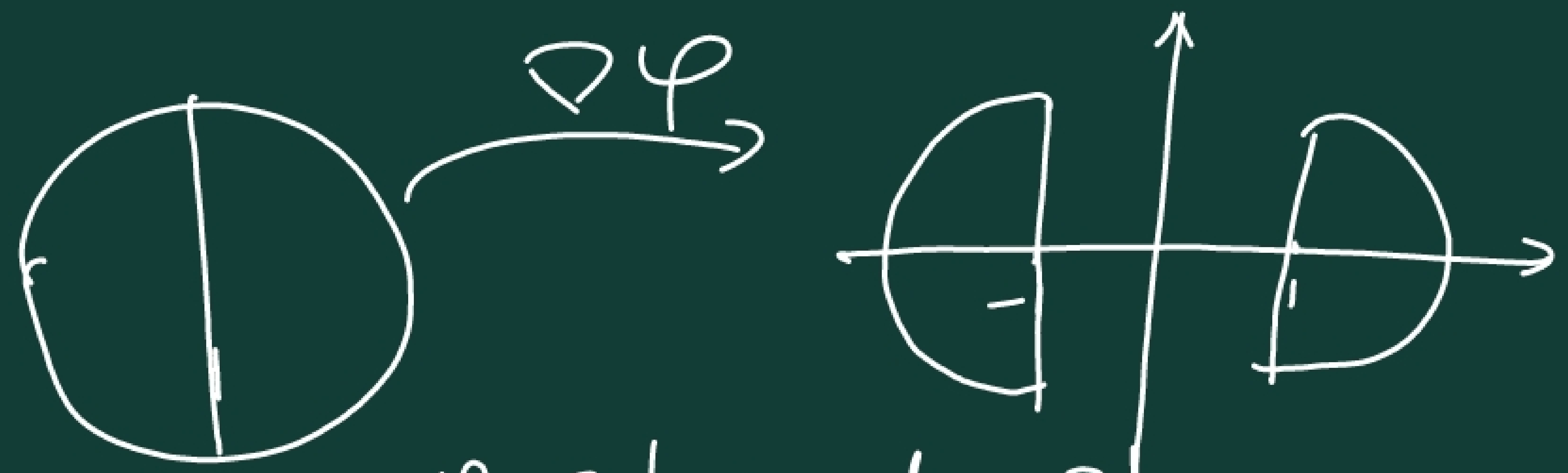
$$\det D^2\psi(x_0) \geq f(x_0, \psi(x_0), \nabla\psi(x_0)).$$

(2) 

$$\det D^2\psi(x_0) \leq f(x_0, \psi(x_0), \nabla\psi(x_0)).$$

An example which is a Brenier soln but not Alexandrov.

$$\varphi(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + |x_1|.$$



φ solves $\det D^2\varphi = 1$ in Brenier sence.

But $\det_H D^2\varphi\big|_{B_1} = 1_{B_1} + \mathcal{H}^1\big|_{\{x_1=0,\ |x_2|\le 1\}}$.