LECTURE NOTES ON S^1-EQUIVARIANT SYMPLECTIC HOMOLOGY

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ABSTRACT. These are a preliminary version of Lecture Notes on positive S^1-equivariant Symplectic homology for the Summer School on Equivariant Symplectic Homology at the Schloß Rauischholzhausen, 23 - 27 July 2018. These notes are essentially a compilation of some parts of the papers [Gut17, GH17, GU17].

This manuscript is written for readers having some background in Floer homology; we recommend for instance the lecture of [AD14] beforehand.
Comments and suggestions are welcome!

CONTENTS

1. Prolegomenon 2
1.1. Motivations 2
1.2. Properties of the equivariant symplectic homology 4
2. Definition of positive S^1-equivariant Symplectic homology 8
2.1. Symplectic homology 9
2.2. Positive symplectic homology 12
2.3. S^1-equivariant symplectic homology 14
2.4. Positive S^1-equivariant symplectic homology 17
2.5. Example: the ball B^{2n} 18
3. Definition of transfer morphisms 21
3.1. Transfer morphisms for (positive) symplectic homology 21
3.2. Transfer morphisms for (positive) S^1-equivariant symplectic homology 25
3.3. Transfer morphisms for generalized Liouville embeddings 26
4. Invariance of symplectic homology 27
4.1. Invariance of the homology of contact fillings 33
5. Properties of positive S^1-equivariant SH 34
5.1. Free homotopy classes 35
5.2. Action filtration 35
5.3. U map 36
5.4. Reeb Orbits 37
5.5. δ map 41
5.6. Scaling 42
5.7. Star-Shaped Domains 43
6. Properties of transfer maps 46
6.1. Action 46

6.2. Functoriality 46
6.3. Commutativity with U 47
6.4. Commutativity with δ 47
Appendix A. Proof of the Reeb orbits property 48
A.1. A lemma on filtered complexes 48
A.2. Construction of $CC_*(X,\lambda)$ 54
Appendix B. Technical Lemmas 57
B.1. My favorite maximum principle 57
B.2. Parenthesis on the homotpy of homotopies Theorem 59
B.3. Parenthesis : A glimpse on signs 60
References 63

1. Prolegomenon

1.1. **Motivations.** Symplectic and contact geometry originated in a mathematical formulation of the classical mechanics of dynamical systems with finitely many degrees of freedom. The objects studied are smooth manifolds with an additional structure, *symplectic* in the even-dimensional case and *contact* in the odd-dimensional one. One of the most prominent feature of symplectic and contact geometry is that rigidity and flexibility phenomena coexist. Flexibility is illustrated in Darboux's theorem (locally all symplectic, respectively all contact, manifolds are "the same") and in various h-principles. Rigidity is illustrated in Gromov's non-squeezing theorem (which is at the origin of symplectic topology) stating that one can symplectically embedd a ball in a cylinder if and only if the radius of the ball is less than that of the cylinder. The comprehension of when rigidity or flexibility occurs and what happens at the "boundary" between those two phenomena is a central question of symplectic and contact geometry/topology.

A *symplectic manifold* is a manifold endowed with a closed, non-degenerate 2-form; this manifold is therefore even-dimensional. A *contact manifold* is an odd-dimensional manifold $(2n-1)$ endowed with a contact structure, i.e. a codimension 1 distribution ξ having a "maximal non-integrability" property. If we write locally the distribution as kernel of a 1-form, $\xi = \operatorname{Ker}\alpha$, the condition is that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing; such a 1-form α is called a *contact form*. If α can be defined globally, the contact structure is said to be co-oriented and the corresponding *Reeb field* is the unique vector field satisfying $\iota(R_\alpha)d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

1.1.1. *Dynamics of the Reeb field.* The Reeb vector field never vanishes (since $\alpha(R_\alpha) = 1$); hence its flow does not have any fixed point. Periodic orbits are thus the most noticeable objects thereof. If α is a contact form on M, $f\alpha$ is also a contact form for any positive function $f \in C^\infty(M,\mathbb{R})$. There are thus as many Reeb fields on a contact manifold M as there are positive smooth functions on M. There is, conjecturally, a strong rigidity stating that if M is compact, each of those Reeb field admit a periodic orbit. It is expressed as Weinstein conjecture, a central question in contact geometry.

Conjecture 1.1 (Weinstein). Every contact form on a compact contact manifold carries at least one periodic Reeb orbit.

The Weinstein conjecture was proven in dimension three by Taubes in 2007 [Tau07]. Taubes' result was later improved by Cristofaro-Gardiner and Hutchings [CGH16] who proved that every contact form on a compact contact manifold of dimension three carries at least two geometrically distinct periodic Reeb orbits. Recently, Cristofaro-Gardiner, Hutchings and Pomerleano [CGHP17] have proven that, modulo assumptions[1], every contact form on a compact contact manifold of dimension three carries either two or infinitely many geometrically distinct periodic Reeb orbits. This last result does not generalize to higher dimensions since Albers, Geiges and Zehmisch [AGZ18] constructed examples, in all dimensions greater than five, of contact forms on compact connected contact manifolds carrying an arbitrarily large (but finite) number of geometrically distinct periodic Reeb orbits.
Those results enhanced the pertinence of the following question:

Question 1.2. Given a contact manifold, what is the lower bound on the number of geometrically distinct periodic Reeb orbits and what is the topological (or analytic) significance of that bound?

This question is actually still open for the sphere with standard contact structure in $\mathbb{R}^{2n}$. The standard contact structure on the sphere S^{2n-1} is defined as the kernel of the 1-form $\alpha_0\big|_{S^{2n-1}} := \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)\big|_{S^{2n-1}}$.

Lemma-Remark 1.3. The study of the Reeb field on all star-shaped[2] hypersurfaces is equivalent to the study of the Reeb field for all contact forms defining the standard contact structure on the sphere S^{2n-1}.

If the star-shaped hypersurface is a regular level set of a smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$, the Reeb field is a multiple of the Hamiltonian field.

Conjecture 1.4. Every star-shaped hypersurface in $\mathbb{R}^{2n}$ carries at least n geometrically distinct periodic Reeb orbits.

1.1.2. *Symplectic embeddings.* Given two symplectic manifolds, (X, ω_X) and (U, ω_U), a *symplectic embedding* $\varphi : X \hookrightarrow U$ is a smooth embedding of X in U such that $\varphi^\star \omega_U = \omega_X$. A general problem in symplectic topology is to classify the symplectic embeddings between two symplectic manifolds X and U of the same dimension. This problem originated in the non-squeezing theorem of Gromov. Let $B^{2n}(r)$ and $Z^{2n}(R) := B^2(R) \times \mathbb{R}^{2n-2}$ denote the ball of radius r, respectively the cylinder of radius R in $\mathbb{R}^{2n}$.

[1]The assumptions are that the contact form is non-degenerate and the first Chern class of the contact structure is torsion

[2]by star-shaped, I mean that the radial vector field is everywhere transverse to the boundary

Theorem 1.5 (Gromov's non-squeezing)**.** The ball $B^{2n}(r)$ symplectically embedds in $Z^{2n}(R)$ if and only if $r \leqslant R$.

Even simple questions on embeddings are completely open: When can you symplectically embedd X into U and, if you can, in how many different ways?

A tool allowing to approach all the above questions would be great; this is precisely what (positive) S^1-equivariant symplectic homology is!

1.2. **Properties of the equivariant symplectic homology.** (Positive) symplectic homology was developed by Viterbo [Vit99], using works of Cieliebak, Floer, and Hofer [FH94, CFH95]. The S^1-equivariant version of (positive) symplectic homology was originally defined by Viterbo [Vit99], and an alternate definition using family Floer homology was given by Bourgeois-Oancea [BO16, §2.2], following a suggestion of Seidel [Sei08]. We will use the family Floer homology definition here, because it is more amenable to computations. We follow the treatment in [Gut17], with some minor tweaks which do not affect the results.

Let (X, λ) be a Liouville domain, so that X is a compact smooth manifold with boundary and $\lambda \in \Omega^1(X)$ has the properties that $d\lambda$ is non-degenerate and that $\lambda|_{\partial X}$ is a contact form. We say that (X, λ) is *non-degenerate* if the linearized return map of the Reeb flow at each closed Reeb orbit on ∂X, acting on the contact hyperplane $\ker \lambda$, does not have 1 as an eigenvalue. We will also assume that the first Chern class of TX vanishes on $\pi_2(X)$.

In this situation, for each $L \in \mathbb{R}$ we have an L-filtered positive S^1-equivariant symplectic homology, $SH^{S^1,+,L}(X, \lambda)$, which will be defined properly in §2.4. To simplify notation, we often denote $SH^{S^1,+,L}(X, \lambda)$ by $CH^L(X, \lambda)$ below[3]. These are $\mathbb{Q}$-vector[4] spaces that come equipped with maps $\imath_{L_1,L_2} \colon CH^{L_1}(X, \lambda) \to CH^{L_2}(X, \lambda)$ for $L_1 \leqslant L_2$ such that $\imath_{L,L}$ is the identity and $\imath_{L_2,L_3} \circ \imath_{L_1,L_2} = \imath_{L_1,L_3}$.[5] The assumption on $c_1(TX)$ implies that the $CH^L(X, \lambda)$ are $\mathbb{Z}$-graded. The (unfiltered) positive S^1-equivariant symplectic homology of (X, λ) is $CH(X, \lambda) = \varinjlim_L CH^L(X, \lambda)$ where the direct limit is constructed using the maps $\imath_{L_1,L_2}$.

Proposition 1.6. The positive S^1-equivariant symplectic homology $CH(X, \lambda)$ has the following properties:

(Free homotopy classes): $CH(X, \lambda)$ has a direct sum decomposition

$$CH(X, \lambda) = \bigoplus_{\Gamma} CH(X, \lambda, \Gamma)$$

[3]The reason for this notation is that positive S^1-equivariant symplectic homology can be regarded as a substitute for linearized contact homology which can be defined without transversality difficulties [BO16, §3.2].

[4]It is also possible to define positive S^1-equivariant symplectic homology with integer coefficients. However the torsion in the latter is not relevant to the applications explained here, and it will simplify our discussion to discard it.

[5]Warning: In [GH17] the map that we denote by $\imath_{L_1,L_2}$ is denoted by $\imath_{L_2,L_1}$.

where Γ ranges over free homotopy classes of loops in X. We let $CH(X,\lambda,0)$ denote the summand corresponding to contractible loops in X.

(Action filtration): For each $L\in\mathbb{R}$, there is a $\mathbb{Q}$-module $CH^L(X,\lambda,\Gamma)$ which is an invariant of (X,λ,Γ). If $L_1<L_2$, then there is a well-defined map

$$\iota_{L_1,L_2} : CH^{L_1}(X,\lambda,\Gamma) \longrightarrow CH^{L_2}(X,\lambda,\Gamma). \tag{1.1}$$

These maps form a directed system, and we have the direct limit

$$\lim_{L\to\infty} CH^L(X,\lambda,\Gamma) = CH(X,\lambda,\Gamma).$$

We denote the resulting map $CH^L(X,\lambda,\Gamma)\to CH(X,\lambda,\Gamma)$ by ι_L. We write $CH^L(X,\lambda)=\bigoplus_\Gamma CH^L(X,\lambda,\Gamma)$.

(U map): There is a distinguished map

$$U : CH(X,\lambda,\Gamma) \longrightarrow CH(X,\lambda,\Gamma),$$

which respects the action filtration in the following sense: For each $L\in\mathbb{R}$ there is a map

$$U_L : CH^L(X,\lambda,\Gamma) \longrightarrow CH^L(X,\lambda,\Gamma).$$

If $L_1<L_2$ then $U_{L_2}\circ\iota_{L_1,L_2}=\iota_{L_1,L_2}\circ U_{L_1}$. The map U is the direct limit of the maps U_L, i.e.

$$\iota_L\circ U_L = U\circ\iota_L. \tag{1.2}$$

(Reeb Orbits): Assume as above that (X,λ) is a non-degenerate Liouville domain with $c_1(TX)|_{\pi_2(X)}=0$. There is an $\mathbb{R}$-filtered chain complex $\big(CC_*(X,\lambda),\partial\big)$, freely generated over $\mathbb{Q}$ by the good[6] Reeb orbits of $\lambda|_{\partial X}$ with the generator corresponding to a Reeb orbit γ having filtration level equal to the action $\int_\gamma\lambda$ and grading equal to the Conley-Zehnder index of γ, such that for each $k\in\mathbb{Z}$ and $L\in\mathbb{R}$ the space $CH_k^L(X,\lambda)$ is the kth homology of the subcomplex $CC_*^L(X,\lambda)$ of $CC_*(X,\lambda)$ consisting of elements with filtration level at most L, and such that for $L_1\leqslant L_2$ the image of the map $\iota_{L_1,L_2}\colon CH_k^{L_1}(X,\lambda)\to CH_k^{L_2}(X,\lambda)$ is isomorphic to the image of the inclusion-induced map $H_k\big(CC_*^{L_1}(X,\lambda)\big)\to H_k\big(CC_*^{L_2}(X,\lambda)\big)$.

Moreover, the boundary operator ∂ on $CC_*(X,\lambda)$ strictly decreases filtration, in the sense that if $x\in CC_*^L(X,\lambda)$ then there is $\varepsilon>0$ such that $\partial x\in CC_*^{L-\varepsilon}(X,\lambda)$.

(δ map): There is a distinguished map

$$\delta : CH(X,\lambda,\Gamma)\longrightarrow H_*(X,\partial X;\mathbb{Q})\otimes H_*(BS^1;\mathbb{Q})$$

which vanishes whenever $\Gamma\neq 0$.

[6]Recall that a Reeb orbit γ is bad if it is an even degree multiple cover of another Reeb orbit γ' such that the Conley-Zehnder indices of γ and γ' have opposite parity. Otherwise, γ is good.

(Scaling): If r is a positive real number, then there are canonical isomorphisms

$$CH(X,\lambda,\Gamma) \xrightarrow{\simeq} CH(X,r\lambda,\Gamma),$$
$$CH^L(X,\lambda,\Gamma) \xrightarrow{\simeq} CH^{rL}(X,r\lambda,\Gamma)$$

which commute with all of the above maps.

(Star-Shaped Domains): If X is a nice star-shaped domain in $\mathbb{R}^{2n}$ and λ_0 is the restriction of the standard Liouville form $\lambda_0 = \frac{1}{2}\sum_{i=1}^n (x_i\,dy_i - y_i\,dx_i)$, then:

(i): $CH(X,\lambda_0)$ and $CH^L(X,\lambda_0)$ have canonical $\mathbb{Z}$ gradings. With respect to this grading, we have

$$CH_*(X,\lambda_0) \simeq \begin{cases} \mathbb{Q}, & \text{if } * \in n+1+2\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

(ii): The map δ sends a generator of $CH_{n-1+2k}(X,\lambda_0)$ to a generator of $H_{2n}(X,\partial X;\mathbb{Q})$ tensor a generator of $H_{2k-2}(BS^1;\mathbb{Q})$.

(iii): The U map has degree -2 and is an isomorphism

$$CH_*(X,\lambda_0) \xrightarrow{\simeq} CH_{*-2}(X,\lambda_0),$$

except when $* = n+1$.

(iv): If $\lambda_0|_{\partial X}$ is nondegenerate and has no Reeb orbit γ with $\mathscr{A}(\gamma) \in (L_1, L_2]$ and $\mathrm{CZ}(\gamma) = n-1+2k$, then the map

$$\imath_{L_2,L_1} : CH^{L_1}_{n-1+2k}(X,\lambda_0) \to CH^{L_2}_{n-1+2k}(X,\lambda_0)$$

is surjective.

Now suppose that (X',λ') is another nondegenerate Liouville domain and $\varphi : (X,\lambda) \to (X',\lambda')$ is a generalized Liouville embedding (see Definition 3.5) with $\varphi(X) \subset \mathrm{int}(X')$. One can then define a **transfer morphism**

$$\Phi : CH(X',\lambda') \longrightarrow CH(X,\lambda),$$

see §3.

Proposition 1.7. The transfer morphism Φ has the following properties:

(Action): Φ respects the action filtration in the following sense: For each $L \in \mathbb{R}$ there are distinguished maps

$$\Phi^L : CH^L(X',\lambda') \longrightarrow CH^L(X,\lambda)$$

such that if $L_1 < L_2$ then

$$\Phi^{L_2} \circ \imath_{L_2,L_1} = \imath_{L_2,L_1} \circ \Phi^{L_1}, \tag{1.4}$$

and Φ is the direct limit of the maps Φ^L, i.e.

$$\imath_L \circ \Phi^L = \Phi \circ \imath_L. \tag{1.5}$$

(Functoriality): The transfer map is functorial in the sense that if (X_1, λ_1), (X_2, λ_2), and (X_3, λ_3) are Liouville domains domains and if $\phi : X_1 \hookrightarrow X_2$ and $\psi : X_2 \hookrightarrow X_3$ are either generalized Liouville embeddings or isomorphisms of Liouville domains, then the following diagram is commutative:

$$CH^L(X_3,\lambda_3) \xrightarrow{\Phi^L_\psi} CH^L(X_2,\lambda_2) \xrightarrow{\Phi^L_\phi} CH^L(X_1,\lambda_1), \qquad \Phi^L_{\psi\circ\phi} : CH^L(X_3,\lambda_3) \to CH^L(X_1,\lambda_1). \tag{1.6}$$

(Commutativity with U): For each $L \in \mathbb{R}$, the diagram

$$\begin{array}{ccc} CH^L(X',\lambda') & \xrightarrow{\Phi^L} & CH^L(X,\lambda) \\ \downarrow U^L & & \downarrow U^L \\ CH^L(X',\lambda') & \xrightarrow{\Phi^L} & CH^L(X,\lambda) \end{array} \tag{1.7}$$

commutes.

(Commutativity with δ): The diagram

$$\begin{array}{ccc} CH(X',\lambda') & \xrightarrow{\Phi} & CH(X,\lambda) \\ \downarrow \delta & & \downarrow \delta \\ H_*(X',\partial X';\mathbb{Q})\otimes H_*(BS^1;\mathbb{Q}) & \xrightarrow{\rho\otimes 1} & H_*(X,\partial X;\mathbb{Q})\otimes H_*(BS^1;\mathbb{Q}) \end{array} \tag{1.8}$$

commutes. Here $\rho : H_*(X', \partial X'; \mathbb{Q}) \to H_*(X, \partial X; \mathbb{Q})$ denotes the composition

$$H_*(X',\partial X';\mathbb{Q}) \longrightarrow H_*(X',X'\backslash\varphi(\text{int}(X));\mathbb{Q}) \xrightarrow{\simeq} H_*(\varphi(X),\varphi(\partial X);\mathbb{Q}) = H_*(X,\partial X;\mathbb{Q})$$

where the first map is the map on relative homology induced by the triple $(X', X'\backslash\varphi(\text{int}(X)), \partial X')$, and the second map is excision.

We shall see applications of this homology to Conjecture 1.4, creation of symplectic capacities for Liouville domains (those are obstructions for existence of symplectic embeddings) and lastly, we to a notion of equivalence of symplectic embeddings and existence of non-equivalent embeddings. But first let's define CH and established its properties.

2. Definition of positive S^1-equivariant Symplectic homology

We will only consider (positive, S^1-equivariant) symplectic homology for Liouville domains, even though it can be defined for more general compact symplectic manifolds with contact-type boundary. We restrict to Liouville domains in order to be able to define transfer morphisms.

Definition 2.1. A *Liouville domain* is a pair (X, λ) where X is a compact manifold with boundary $\partial X = Y$, λ is a 1-form such that $\omega := d\lambda$ is symplectic on X and the Liouville vector field Z defined by $\iota(Z)\omega = \lambda$ points strictly outwards along Y.

We denote by α the *contact* 1*-form* on Y which is the restriction of λ to Y: $\alpha = \lambda|_Y$. We denote by ξ the *contact structure* on Y defined by α, i.e $\xi := \ker\alpha$. The *Reeb vector field* R_α is the vector field on Y defined by :

$$\begin{cases} \iota(R_\alpha)d\alpha = 0 \\ \alpha(R_\alpha) = 1. \end{cases}$$

The *action spectrum* of (Y,α) is the set of all periods of the Reeb vector field

$$\mathrm{Spec}(Y,\alpha) := \{T \in \mathbb{R}^+ \mid \exists\gamma \text{ periodic orbit of } R_\alpha \text{ of period } T\}.$$

The *symplectic completion* of (X,λ) is the symplectic manifold defined by

$$\widehat{X} := X\bigcup_G(\mathbb{R}^+\times Y) := \big(X \sqcup ([-\delta,+\infty)\times Y)\big)/\sim_G$$

with the symplectic form

$$d\widehat{\lambda} := \begin{cases} d\lambda & \text{on } X \\ d(e^\rho\alpha) & \text{on } [-\delta,+\infty)\times Y \end{cases}.$$

The equivalence $\sim_G$, between a neighbourhood U of Y in X and $[-\delta,0]\times Y$, is defined by the diffeomorphism

$$G : [-\delta,0]\times Y \to U : \quad (\rho,p) \mapsto \varphi^Z_\rho(p)$$

where φ^Z is the flow of the Liouville vector field Z. This is always possible since

$$G^\star\omega = e^\rho(d\alpha + d\rho\wedge\alpha) = d(e^\rho\alpha).$$

Observe indeed that $G_{\star(\rho,y)}\left(\frac{\partial}{\partial\rho}\right) = Z_{G(\rho,y)}, \quad G_{\star(\rho,y)}\left(A_y\right) = \left(\varphi^Z_\rho\right)_{\star y}A_y$. Since $\lambda(Z) = (\iota(Z)\omega)(Z) = 0$, we also have $\mathscr{L}_Z\lambda = \lambda$ and $\left(\varphi^Z_\rho\right)^\star\lambda = e^\rho\lambda$. Hence, $\forall A_y, B_y \in T_yY$:

$$\begin{aligned}
(G^\star\omega)_{(\rho,y)}\left(\frac{\partial}{\partial\rho},A_y\right) &= \omega_{\varphi^Z_\rho y}\left(Z_{G(\rho,y)},\left(\varphi^Z_\rho\right)_{\star y}A_y\right) \\
&= \left(\left(\varphi^Z_\rho\right)^\star\omega\right)_y(Z_y,A_y) = e^\rho d\lambda_y(Z_y,A_y) = e^\rho\left(\mathscr{L}_Z\lambda\right)_y(A_y) \\
&= e^\rho(d\alpha + d\rho\wedge\alpha)\left(\frac{\partial}{\partial\rho},A_y\right) \\
(G^\star\omega)_{(\rho,y)}\left(A_y,B_y\right) &= \omega_{\varphi^Z_\rho y}\left(\left(\varphi^Z_\rho\right)_{\star y}A_y,\left(\varphi^Z_\rho\right)_{\star y}B_y\right) \\
&= \left(\left(\varphi^Z_\rho\right)^\star\omega\right)_y(A_y,B_y) = e^\rho d\lambda_y(A_y,B_y) = e^\rho d\alpha_y(A_y,B_y) \\
&= e^\rho(d\alpha + d\rho\wedge\alpha)\left(A_y,B_y\right).
\end{aligned}$$

Remark 2.2. We refer to the wonderful blog from Chris Wendl, [Wen15], as to why the completion is $\mathbb{R}^+\times Y$ and not $Y\times\mathbb{R}^+$. I also confess that I am using "wrong" signs in the following for the Hamiltonian vector field...

2.1. **Symplectic homology.** Let (X,λ) be a Liouville domain with boundary Y. Let R_λ denote the Reeb vector field associated to λ on Y. Below, let $\operatorname{Spec}(Y,\lambda)$ denote the set of periods of Reeb orbits, and let $\varepsilon = \frac{1}{2}\min\operatorname{Spec}(Y,\lambda)$.

Recall that the completion $(\widehat{X},\widehat{\lambda})$ of (X,λ) is defined by

$$\widehat{X} := X \cup \big([0,\infty)\times Y\big) \quad \text{and} \quad \widehat{\lambda} := \begin{cases} \lambda & \text{on } X, \\ e^\rho \lambda|_Y & \text{on } [0,\infty)\times Y \end{cases}$$

where ρ denotes the $[0,\infty)$ coordinate. Write $\widehat{\omega} = d\widehat{\lambda}$. Consider a 1-periodic Hamiltonian on $\widehat{X}$, i.e. a smooth function

$$H : S^1 \times \widehat{X} \longrightarrow \mathbb{R}$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. Such a function H determines a vector field X_H^θ on $\widehat{X}$ for each $\theta \in S^1$, defined by $\widehat{\omega}(X_H^\theta,\cdot) = dH(\theta,\cdot)$. Let $\mathscr{P}(H)$ denote the set of 1-periodic orbits of X_H, i.e. smooth maps $\gamma : S^1 \to \widehat{X}$ satisfying the equation $\gamma'(\theta) = X_H^\theta\big(\gamma(\theta)\big)$.

Definition 2.3. An **admissible Hamiltonian** is a smooth function $H : S^1\times\widehat{X} \to \mathbb{R}$ satisfying the following conditions:

- **(1):** The restriction of H to $S^1 \times X$ is negative, autonomous (i.e. S^1-independent), and C^2-small (so that there are no non-constant 1-periodic orbits). Furthermore,

$$H > -\varepsilon \tag{2.1}$$

  on $S^1 \times X$.
- **(2):** There exists $\rho_0 \geqslant 0$ such that on $S^1 \times [\rho_0,\infty)\times Y$ we have

$$H(\theta,\rho,y) = \beta e^\rho + \beta' \tag{2.2}$$

  with $0 < \beta \notin \operatorname{Spec}(Y,\lambda)$ and $\beta' \in \mathbb{R}$. The constant β is called the **limiting slope** of H.
- **(3):** There exists a small, strictly convex, increasing function $h : [1,e^{\rho_0}] \to \mathbb{R}$ such that on $S^1\times[0,\rho_0]\times Y$, the function H is C^2-close to the function sending $(\theta,\rho,x)\mapsto h(e^\rho)$. The precise sense of "small" and "close" that we need here is explained in Remarks 2.4 and 2.8.
- **(4):** The Hamiltonian H is nondegenerate, i.e. all 1-periodic orbits of X_H are nondegenerate.

We denote the set of admissible Hamiltonians by $\mathscr{H}_{\text{std}}$.

Remark 2.4. Condition (1) implies that the only 1-periodic orbits of X_H in X are constants; they correspond to critical points of H.

The significance of condition (2) is as follows. On $S^1\times[0,\infty)\times Y$, for a Hamiltonian of the form $H_1(\theta,\rho,y) = h_1(e^\rho)$, we have

$$X_{H_1}^\theta(\rho,y) = -h_1'(e^\rho)R_\lambda(y).$$

Hence for such a Hamitonian H_1 with h_1 increasing, a 1-periodic orbit of X_{H_1} maps to a level $\{\rho\}\times Y$, and the image of its projection to Y is the image of a (not necessarily simple) periodic Reeb orbit of period $h_1'(e^\rho)$. In particular, condition (2) implies that there is no 1-periodic orbit of X_H in $[\rho_0, \infty) \times Y$.

Condition (3) ensures that for any non-constant 1-periodic orbit γ_H of X_H, there exists a (not necessarily simple) periodic Reeb orbit γ of period $T < \beta$ such that the image of γ_H is close to the image of γ in $\{\rho\} \times Y$ where $T = h'(e^\rho)$.

Definition 2.5. An S^1-family of almost complex structures $J : S^1 \to \operatorname{End}(T\widehat{X})$ is **admissible** if it satisfies the following conditions:

- J^θ is $\widehat{\omega}$-compatible for each $\theta \in S^1$.
- There exists $\rho_1 \geqslant 0$ such that on $[\rho_1, \infty) \times Y$, the almost complex structure J^θ does not depend on θ, is invariant under translation of ρ, sends ξ to itself compatibly with $d\lambda$, and satisfies

$$J^\theta(\partial_\rho) = R_\lambda. \tag{2.3}$$

We denote the set of all admissible J by $\mathcal{J}$.

Given $J \in \mathcal{J}$, and $\gamma_-, \gamma_+ \in \mathcal{P}(H)$, let $\widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$ denote the set of maps

$$u : \mathbb{R} \times S^1 \longrightarrow \widehat{X}$$

satisfying Floer's equation

$$\frac{\partial u}{\partial s}(s,\theta) + J^\theta\big(u(s,\theta)\big)\left(\frac{\partial u}{\partial \theta}(s,\theta) - X_H^\theta\big(u(s,\theta)\big)\right) = 0 \tag{2.4}$$

as well as the asymptotic conditions

$$\lim_{s\to\pm\infty} u(s,\cdot) = \gamma_\pm.$$

If J is generic and $u \in \widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$, then $\widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$ is a manifold near u whose dimension is the Fredholm index of u defined by

$$\operatorname{ind}(u) = \mathrm{CZ}_\tau(\gamma_+) - \mathrm{CZ}_\tau(\gamma_-).$$

Here CZ_τ denotes the Conley-Zehnder index computed using trivializations τ of $\gamma_\pm^\star T\widehat{X}$ that extend to a trivialization of $u^\star T\widehat{X}$. Note that $\mathbb{R}$ acts on $\widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$ by translation of the domain; we denote the quotient by $\mathcal{M}(\gamma_-, \gamma_+; J)$.

Definition 2.6. Let $H \in \mathcal{H}_{\mathrm{std}}$, and let $J \in \mathcal{J}$ be generic. Define the Floer chain complex $(CF(H,J), \partial)$ as follows. The chain module $CF(H,J)$ is the free $\mathbb{Q}$-module[7] generated by the set of 1-periodic orbits $\mathcal{P}(H)$. If $\gamma_-, \gamma_+ \in \mathcal{P}(H)$, then the coefficient of γ_+ in $\partial\gamma_-$ is obtained by counting Fredholm index 1 points in $\mathcal{M}(\gamma_-, \gamma_+; J)$ with signs determined by a system of coherent orientations as in [FH93]. (The chain complexes for different choices of coherent orientations are canonically isomorphic.)

[7]It is also possible to use $\mathbb{Z}$ coefficients here, but we will use $\mathbb{Q}$ coefficients in order to later establish the Reeb Orbits property in Proposition 1.6, which leads to the Reeb Orbits property of the capacities c_k. In special cases when the Conley-Zehnder index of a 1-periodic orbit is unambiguously defined, for example when all 1-periodic orbits are contractible and $c_1(TX)|_{\pi_2(X)} = 0$, the chain complex is graded by minus the Conley-Zehnder index.

Let $HF(H,J)$ denote the homology of the chain complex $(CF(H,J),\partial)$. Given H, the homologies for different choices of generic J are canonically isomorphic to each other, so we can denote this homology simply by $HF(H)$.

The construction of the above canonical isomorphisms is a special case of the following more general construction. Given two admissible Hamiltonians $H_1, H_2 \in \mathscr{H}_{\mathrm{std}}$, write $H_1 \leqslant H_2$ if $H_1(\theta,x) \leqslant H_2(\theta,x)$ for all $(\theta,x) \in S^1 \times \widehat{X}$. In this situation, one defines a *continuation morphism* $HF(H_1) \to HF(H_2)$ as follows; cf. [Gut17, Thm. 4.5] and the references therein. Choose generic $J_1, J_2 \in \mathscr{J}$ so that the chain complexes $CF(H_i,J_i)$ are defined for $i = 1,2$. Choose a generic homotopy $\{(H_s,J_s)\}_{s\in\mathbb{R}}$ such that H_s satisfies equation (2.2) for some β,β' depending on s; $J_s \in \mathscr{J}$ for each $s \in \mathbb{R}$; $\partial_s H_s \geqslant 0$; $(H_s,J_s) = (H_1,J_1)$ for $s << 0$; and $(H_s,J_s) = (H_2,J_2)$ for $s >> 0$. One then defines a chain map $CF(H_1,J_1) \to CF(H_2,J_2)$ as a signed count of Fredholm index 0 maps $u : \mathbb{R} \times S^1 \to \widehat{X}$ satisfying the equation

$$\frac{\partial u}{\partial s} + J_s^\theta \circ u\left(\frac{\partial u}{\partial \theta} - X_{H_s}^\theta \circ u\right) = 0 \tag{2.5}$$

and the asymptotic conditions $\lim_{s\to-\infty} u(s,\cdot) = \gamma_1$ and $\lim_{s\to\infty} u(s,\cdot) = \gamma_2$. The induced map on homology gives a well-defined map $HF(H_1) \to HF(H_2)$. If $H_2 \leqslant H_3$, then the continuation map $HF(H_1) \to HF(H_3)$ is the composition of the continuation maps $HF(H_1) \to HF(H_2)$ and $HF(H_2) \to HF(H_3)$.

Definition 2.7. We define the *symplectic homology* of (X,λ) to be the direct limit

$$SH(X,\lambda) := \varinjlim_{H\in\mathscr{H}_{\mathrm{adm}}} HF(H)$$

with respect to the partial order $\leqslant$ and continuation maps defined above.

Direct limits. Let A_i, $i \in I$ be abelian groups and let for all pair $i,j \in I$, $\varphi_{i,j}$ be the homomorphism $\varphi_{i,j} : A_i \to A_j$ such that $\varphi_{i,i} = \mathrm{Id}$ and for all triple $i \leqslant j \leqslant k$, $\varphi i,k = \varphi_{j,k} \circ \varphi_{i,j}$. The set $\{A_i, \varphi_{i,j}\}$ is called a *directed system* of groups. The *direct limit*, $\lim_{\to} A_i$, of a directed system is the unique group L, up to isomorphism having the following universal property: *There exists maps $\varphi_i : A_i \to L$ such that for all $i < j$, $\varphi_i = \varphi_j \circ \varphi_{i,j}$ and if C is an abelian group together with maps $\tau_i : A_i \to C$ such that $\tau_i = \tau_j \circ \varphi_{i,j}$ for all $i < j$, then there exist a unique homomorphism $\tau : L \to C$ such that $\tau_i = \tau \circ \varphi_i$ for all i.*

2.2. **Positive symplectic homology.**

Positive symplectic homology is a modification of symplectic homology in which constant 1-periodic orbits are discarded.

To explain this, let $H : S^1 \times \widehat{X} \to \mathbb{R}$ be a Hamiltonian in $\mathscr{H}_{\mathrm{std}}$. The *Hamiltonian action* functional $\mathscr{A}_H : C^\infty(S^1,\widehat{X}) \to \mathbb{R}$ is defined by

$$\mathscr{A}_H(\gamma) := -\int_{S^1} \gamma^\star \widehat{\lambda} - \int_{S^1} H\big(\theta,\gamma(\theta)\big)d\theta.$$

If $J \in \mathscr{J}$, then the differential on the chain complex $(CF(H,J),\partial)$ decreases the Hamiltonian action $\mathscr{A}_H$. As a result, for any $L \in \mathbb{R}$, we have a subcomplex $CF^{\leqslant L}(H,J)$ of $CF(H,J)$, generated by the 1-periodic orbits with Hamiltonian action less than or equal to L.

To see what this subcomplex can look like, note that the 1-periodic orbits of $H \in \mathscr{H}_{\text{std}}$ fall into two classes: (i) constant orbits corresponding to critical points in X, and (ii) non-constant orbits contained in $[0,\rho_0] \times Y$.

If x is a critical point of H on X, then the action of the corresponding constant orbit is equal to $-H(x)$. By (2.1), this is less than ε.

By Remark 2.4, a non-constant 1-periodic orbit of X_H is close to a 1-periodic orbit of $-h'(e^\rho)R_\lambda$ located in $\{\rho\} \times Y$ for $\rho \in [0,\rho_0]$ with $h'(e^\rho) \in \operatorname{Spec}(Y,\lambda)$. The Hamiltonian action of the latter loop is given by

$$-\int_{S^1} e^\rho \lambda(-h'(e^\rho)R_\lambda) d\theta - \int_{S^1} h(e^\rho) d\theta = e^\rho h'(e^\rho) - h(e^\rho). \tag{2.6}$$

Since h is strictly convex, the right hand side is a strictly increasing function of ρ.

Remark 2.8. In Definition 2.3, we assume that h is sufficiently small so that the right hand side of (2.6) is close to the period $h'(e^\rho)$, and in particular greater than ε. We also assume that H is sufficiently close to $h(e^\rho)$ on $S^1 \times [0,\rho_0] \times Y$ so that the Hamiltonian actions of the 1-periodic orbits are well approximated by the right hand side of (2.6), so that:

- **(i):** The Hamiltonian action of every 1-periodic orbit of X_H corresponding to a critical point on X is less than ε; and the Hamiltonian action of every other 1-periodic orbit is greater than ε.
- **(ii):** If γ is a Reeb orbit of period $T < \beta$, and if γ' is a 1-periodic orbit of X_H in $[0,\rho_0] \times Y$ associated to γ, then

  $$|\mathscr{A}_H(\gamma') - T| < \min\left\{\beta^{-1}, \tfrac{1}{3}\operatorname{gap}(\beta)\right\}.$$

  Here $\operatorname{gap}(\beta)$ denotes the minimum difference between two elements of $\operatorname{Spec}(Y,\lambda)$ that are less than β.

We can now define positive symplectic homology.

Definition 2.9. Let (X,λ) be a Liouville domain, let H be a Hamiltonian in $\mathscr{H}_{\text{std}}$, and let $J \in \mathscr{J}$.

Consider the quotient complex

$$CF^+(H,J) := \frac{CF(H,J)}{CF^{\leqslant\varepsilon}(H,J)}.$$

The homology of the quotient complex is independent of J, so we can denote this homology by $HF^+(H)$. More generally, if $H_1 \leqslant H_2$, then the chain map used to define the continuation map $HF(H_1) \to HF(H_2)$ descends to the quotient, since the Hamiltonian action is nonincreasing along a solution of (2.5) when the homotopy is nondecreasing. Thus we obtain a well-defined continuation map $HF^+(H_1) \to HF^+(H_2)$ satisfying the same properties as before.

We now define the *positive symplectic homology* of (X,λ) to be the direct limit

$$SH^+(X,\lambda) := \varinjlim_{H \in \mathscr{H}_{\text{std}}} HF^+(H).$$

Positive symplectic homology can sometimes be better understood using certain special admissible Hamiltonians obtained as follows.

Definition 2.10. [BO09] Let (X,λ) be a Liouville domain. An *admissible Morse-Bott Hamiltonian* is an autonomous Hamiltonian $H:\widehat{X}\to\mathbb{R}$ such that:

(1): The restriction of H to X is a Morse function which is negative and C^2-small (so that the Hamiltonian vector field has no non-constant 1-periodic orbits).

(2): There exists $\rho_0 \geqslant 0$ such that on $[\rho_0,\infty)\times Y$ we have

$$H(\rho,x)=\beta e^{\rho}+\beta'$$

with $0<\beta\notin \operatorname{Spec}(Y,\lambda)$ and $\beta'\in\mathbb{R}$.

(3): On $[0,\rho_0)\times Y$ we have

$$H(\rho,x)=h(e^{\rho})$$

where h is as in Definition 2.3, and moreover $h''-h'>0$.

We denote the set of admissible Morse-Bott Hamiltonians by $\mathscr{H}_{\mathrm{MB}}$.

Given $H\in\mathscr{H}_{\mathrm{MB}}$, each 1-periodic orbit of X_H is either: (i) a constant orbit corresponding to a critical point of H in X, or (ii) a non-constant 1-periodic orbit, with image in $\{\rho\}\times Y$ for $\rho\in(0,\rho_0)$, whose projection to Y has the same image as a Reeb orbit of period $e^{\rho}h'(\rho)$. Since H is autonomous, every Reeb orbit γ with period less than β gives rise to an S^1 family of 1-periodic orbits of X_H, which we denote by S_γ.

An admissible Morse-Bott Hamiltonian as in Definition 2.10 can be deformed into an admissible Hamiltonian as in Definition 2.3, which will be time-dependent and have nondegenerate 1-periodic orbits:

Lemma 2.11. ([CFHW96, Prop. 2.2] and [BO09, Lem. 3.4]) An admissible Morse-Bott Hamiltonian H can be perturbed to an admissible Hamiltonian H' whose 1-periodic orbits consist of the following:

(i): Constant orbits at the critical points of H.

(ii): For each Reeb orbit γ with period less than β, two nondegenerate orbits $\widehat{\gamma}$ and $\check{\gamma}$. Given a trivialization τ of $\xi|\gamma$, their Conley-Zehnder indices are given by $-\mathrm{CZ}_\tau(\widehat{\gamma})=\mathrm{CZ}_\tau(\gamma)+1$ and $-\mathrm{CZ}_\tau(\check{\gamma})=\mathrm{CZ}_\tau(\gamma)$.

Remark 2.12. The references [CFHW96] and [BO09] use the notation $\gamma_{\min}$ instead of $\widehat{\gamma}$, and γ_{Max} instead of $\check{\gamma}$. The motivation is that these orbits are distinguished in their S^1-family as critical points of a perfect Morse function on S^1.

2.3. **S^1-equivariant symplectic homology.** Let (X,λ) be a Liouville domain with boundary Y. We now review how to define the S^1-equivariant symplectic homology $SH^{S^1}(X,\lambda)$, and the positive S^1-equivariant symplectic homology $SH^{S^1,+}(X,\lambda)$.

The S^1-equivariant symplectic homology $SH^{S^1}(X,\lambda)$ is defined as a limit as $N\to\infty$ of homologies $SH^{S^1,N}(X,\lambda)$, where N is a nonnegative integer. To define the latter, fix the perfect Morse function $f_N : \mathbb{C}P^N \to \mathbb{R}$ defined by

$$f_N\left([w^0 : \dots : w^n]\right) = \frac{\sum_{j=0}^N j|w^j|^2}{\sum_{j=0}^N |w^j|^2}.$$

Let $\widetilde{f}_N : S^{2N+1} \to \mathbb{R}$ denote the pullback of f_N to S^{2N+1}. We will consider gradient flow lines of $\widetilde{f_N}$ and f_N with respect to the standard metric on S^{2N+1} and the metric that this induces on $\mathbb{C}P^N$.

Remark 2.13. The family of functions f_N has the following two properties which are needed below. We have two isometric inclusions $i_0, i_1 : \mathbb{C}P^N \to \mathbb{C}P^{N+1}$ defined by $i_0([z_0 : \dots : z_N]) = [z_0 : \dots : z_N : 0]$ and $i_1([z_0 : \dots : z_N]) = [0 : z_0 : \dots : z_N]$. Then:

(1): The images of i_0 and i_1 are invariant under the gradient flow of f_{N+1}.
(2): We have $f_N = f_{N+1} \circ i_0 = f_{N+1} \circ i_1 +$ constant, so that the gradient flow of f_{N+1} pulls back via i_0 or i_1 to the gradient flow of f_N.

Now choose a "parametrized Hamiltonian"

$$H : S^1 \times \widehat{X} \times S^{2N+1} \longrightarrow \mathbb{R} \tag{2.7}$$

which is S^1-invariant in the sense that

$$H(\theta + \varphi, x, \varphi z) = H(\theta, x, z) \qquad \forall \theta, \varphi \in S^1 = \mathbb{R}/\mathbb{Z},\ x \in \widehat{X},\ z \in S^{2N+1}.$$

Here the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $S^{2N+1} \subset \mathbb{C}^{N+1}$ is defined by $\varphi \cdot z = e^{2\pi i \varphi} z$.

Definition 2.14. A parametrized Hamiltonian H as above is *admissible* if:

(i): For each $z \in S^{2N+1}$, the Hamiltonian

$$H_z = H(\cdot, \cdot, z) : S^1 \times \widehat{X} \longrightarrow \mathbb{R}$$

satisfies conditions (1), (2), and (3) in Definition 2.3, with β and β' independent of z.
(ii): If z is a critical point of $\widetilde{f}_N$, then the 1-periodic orbits of H_z are non-degenerate.
(iii): H is nondecreasing along downward gradient flow lines of $\widetilde{f}_N$.

Let $\mathscr{P}^{S^1}(\tilde{f}_N, H)$ denote the set of pairs (z, γ), where $z \in S^{2N+1}$ is a critical point of $\tilde{f}_N$, and γ is a 1-periodic orbit of the Hamitonian H_z. Note that S^1 acts freely on the set $\mathscr{P}^{S^1}(\tilde{f}_N, H)$ by

$$\varphi \cdot (z, \gamma) = \left(\varphi \cdot z, \gamma(\cdot - \varphi)\right).$$

If $p = (z, \gamma) \in \mathscr{P}^{S^1}(\tilde{f}_N, H)$, let S_p denote the orbit of (z, γ) under this S^1 action.
Next, choose a generic map

$$J : S^1 \times S^{2N+1} \to \mathscr{J}, \quad (\theta, z) \mapsto J_z^\theta, \tag{2.8}$$

which is S^1-invariant in the sense that

$$J_{\varphi \cdot z}^{\theta + \varphi} = J_z^\theta$$

for all $\varphi,\theta\in S^1$ and $z\in S^{2N+1}$.

Let $p^-=(z^-,\gamma^-)$ and $p^+=(z^+,\gamma^+)$ be distinct elements of $\mathscr{P}^{S^1}(\tilde{f}_N,H)$. Define $\widehat{\mathscr{M}}(S_{p^-},S_{p^+};J)$ to be the set of pairs (η,u), where $\eta:\mathbb{R}\to S^{2N+1}$ and $u:\mathbb{R}\times S^1\to\widehat{X}$, satisfying the following equations:

$$\begin{cases} \dot{\eta}+\vec{\nabla}\tilde{f}_N(\eta)=0, \\ \partial_s u+J^\theta_{\eta(s)}\circ u\big(\partial_\theta u-X_{H^\theta_{\eta(s)}}\circ u\big)=0, \\ \lim_{s\to\pm\infty}\big(\eta(s),u(s,\cdot)\big)\in S_{p^\pm}. \end{cases} \tag{2.9}$$

Here the middle equation is a modification of Floer's equation (2.4) which is "parametrized by η". Note that $\mathbb{R}$ acts on the set $\widehat{\mathscr{M}}(S_{p^-},S_{p^+};J)$ by reparametrization: if $\sigma\in\mathbb{R}$, then

$$\sigma\cdot(\eta,u)=\big(\eta(\cdot-\sigma),u(\cdot-\sigma,\cdot)\big).$$

In addition, S^1 acts on the set $\widehat{\mathscr{M}}(S_{p^-},S_{p^+};J)$ as follows: if $\tau\in S^1$, then

$$\tau\cdot(\eta,u):=\big(\tau\cdot\eta,u(\cdot,\cdot-\tau)\big).$$

Let $\mathscr{M}^{S^1}(S_{p^-},S_{p^+};J)$ denote the quotient of the set $\widehat{\mathscr{M}}(S_{p^-},S_{p^+};J)$ by these actions of $\mathbb{R}$ and S^1.

If J is generic, then $\mathscr{M}^{S^1}(S_{p^-},S_{p^+};J)$ is a manifold near (η,u) of dimension

$$\operatorname{ind}(\eta,u)=(\operatorname{ind}(f_N,z^-)-\mathrm{CZ}_\tau(\gamma^-))-(\operatorname{ind}(f_N,z^+)-\mathrm{CZ}_\tau(\gamma^+))-1.$$

Here $\operatorname{ind}(f_N,z^\pm)$ denotes the Morse index of the critical point $z^\pm$ of f_N, and CZ_τ denotes the Conley-Zehnder index with respect to a trivialization τ of $(\gamma^\pm)^\star T\widehat{X}$ that extends over $u^\star T\widehat{X}$.

Definition 2.15. [BO16, §2.2] Define a chain complex $\Big(CF^{S^1,N}(H,J),\partial^{S^1}\Big)$ as follows. The chain module $CF^{S^1,N}(H,J)$ is the free $\mathbb{Q}$ module[8] generated by the orbits S_p. If S_{p^-},S_{p^+} are two such orbits, then the coefficient of S_{p^+} in $\partial^{S^1}S_{p^-}$ is a signed count of elements (η,u) of $\mathscr{M}^{S^1}(S_{p^-},S_{p^+};J)$ with $\operatorname{ind}(\eta,u)=1$.

We denote the homology of this chain complex by $HF^{S^1,N}(H)$. This does not depend on the choice of J, by the usual continuation argument; one defines continuation chain maps using a modification of (2.9) in which the second line is replaced by an "η-parametrized" version of Floer's continuation equation (2.5).

We now define a partial order on the set of pairs (N,H), where N is a non-negative integer and H is an admissible parametrized Hamiltonian (2.7), as follows. Let $\tilde{i}_0:S^{2N+1}\to S^{2N+3}$ denote the inclusion sending $z\mapsto(z,0)$. (This lifts the inclusion i_0 defined in Remark 2.13.) Then $(N_1,H_1)\leqslant(N_2,H_2)$ if and only if:

- $N_1\leqslant N_2$, and
- $H_1\leqslant(\tilde{i}_0^\star)^{N_2-N_1}H_2$ pointwise on $S^1\times\widehat{X}\times S^{2N_1+1}$.

[8]It is also possible to define $SH^{S^1,+}$, using $\mathbb{Z}$ coefficients, as with SH.

In this case we can define a continuation map $HF^{S^1,N_1}(H_1) \to HF^{S^1,N_2}(H_2)$ using an increasing homotopy from H_1 to $(\widetilde{i_0}^\star)^{N_2-N_1}H_2$ on $S^1 \times \widehat{X} \times S^{2N_1+1}$.

Definition 2.16. Define the S^1*-equivariant symplectic homology*

$$SH_*^{S^1}(X,\lambda) := \varinjlim_{N,H} HF_*^{S^1,N}(H).$$

It is sometimes useful to describe S^1-equivariant symplectic homology in terms of individual Hamiltonians on $S^1 \times \widehat{X}$, rather than S^{2N+1}-families of them, by the following procedure.

Remark 2.17. [Gut14, §2.1.1] Fix an admissible Hamiltonian $H' : S^1 \times \widehat{X} \to \mathbb{R}$ and a nonnegative integer N. Consider a sequence of admissible parametrized Hamiltonians $\{H_k\}_{k=0,\dots,N}$ as in (2.7), where H_k is defined on $S^1 \times \widehat{X} \times S^{2k+1}$, with the following properties:

- For each $k = 0,\dots,N-1$, the pullbacks $\widetilde{i}_0^\star H_{k+1}$ and $\widetilde{i}_1^\star H_{k+1}$ agree with H_k up to a constant. Here $\widetilde{i}_1 : S^{2k+1} \to S^{2k+3}$ denotes the lift of i_1 sending $z \mapsto (0,z)$.
- For each $k = 0,\dots,N$ and each $z \in \mathrm{Crit}(\tilde{f}_k)$, we have

$$H_k(\theta,x,z) = H'\big(\theta - \phi(z), x\big) + c. \tag{2.10}$$

  Here c is a constant depending on k and z; and the map $\phi : \mathrm{Crit}(\tilde{f}_k) \to S^1$ sends a critical point $(0,\dots,0,e^{2\pi i\psi},0,\dots,0) \mapsto \psi$.

Next, choose a sequence of families of almost complex structures $J_k : S^1 \times S^{2k+1} \to \mathscr{J}(\widehat{X})$ for $k = 0,\dots,N$ such that:

- J_k is generic so that the chain complex $\left(CF^{S^1,k}(H_k,J_k), \partial^{S^1}\right)$ is defined.
- $\widetilde{i}_0^\star J_{k+1} = \widetilde{i}_1^\star J_{k+1} = J_k$.

The chain complex $\left(CF^{S^1,N}(H_N,J_N), \partial^{S^1}\right)$ can now be described as follows. By (2.10), we can identify the chain module as

$$CF^{S^1,N}(H_N,J_N) = \mathbb{Q}\{1,u,\dots,u^N\} \otimes_{\mathbb{Q}} CF(H',J_0). \tag{2.11}$$

This identification sends a pair (z,γ), where $z \in \mathrm{Crit}(\widetilde{f}_N)$ is a lift of an index $2k$ critical point of f_N and γ is a reparametrization of a 1-periodic orbit γ' of H', to $u^k \otimes \gamma'$.

Since the sequences $\{H_k\}$ and $\{J_k\}$ respect the inclusions $\widetilde{i}_1$, the differential has the form

$$\partial^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_i(\gamma) \tag{2.12}$$

where the operator φ_i on $CF(H',J_0)$ does not depend on k. In particular, φ_0 is the differential on $CF(H',J_0)$. We can also formally write

$$\partial^{S^1} = \sum_{i=0}^{N} u^{-i} \otimes \varphi_i$$

where it is understood that u^{-i} annihilates terms of the form $u^j \otimes \gamma$ with $i > j$.

The usual continuation arguments show that the homology of this chain complex does not depend on the choice of sequences $\{H_k\}$ and $\{J_k\}$ satisfying the above assumptions. We denote this homology by $HF^{S^1,N}(H')$.

Since in the above construction we assume that the sequences $\{H_k\}$ and $\{J_k\}$ respect the inclusions $\widetilde{i}_0$, it follows that when $N_1 \leqslant N_2$ we have a well-defined map $HF^{S^1,N_1}(H') \to HF^{S^1,N_2}(H')$ induced by inclusion of chain complexes.

As before, if $H'_1 \leqslant H'_2$, then there is a continuation map $HF^{S^1,N}(H'_1) \to HF^{S^1,N}(H'_2)$ satisfying the usual properties.

As in [BO16, §2.3], we now have:

Proposition 2.18. The S^1-equivariant homology of (X,λ) is given by

$$SH_*^{S^1}(X,\lambda) = \varinjlim_{N\in\mathbb{N},\, H'\in\mathscr{H}_{\mathrm{std}}} HF^{S^1,N}(H').$$

2.4. **Positive S^1-equivariant symplectic homology.** Like symplectic homology, S^1-equivariant symplectic homology also has a positive version in which constant 1-periodic orbits are discarded.

Definition 2.19. Let $H : S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$ be an admissible parametrized Hamiltonian. The *parametrized action functional* $\mathscr{A}_H : C^\infty(S^1,\widehat{X}) \times S^{2N+1} \longrightarrow \mathbb{R}$ is defined by

$$\mathscr{A}_H(z,\gamma) := -\int_\gamma \widehat{\lambda} - \int_{S^1} H\big(\theta,\gamma(\theta),z\big)\, d\theta. \tag{2.13}$$

Lemma 2.20. If H is an admissible parametrized Hamiltonian, and if J is a generic S^1-invariant family of almost complex structures as in (2.8), then the differential ∂^{S^1} on $CF^{S^1,N}(H,J)$ does not increase the parametrized action (2.13).

Proof. Given a solution (η,u) to the equations (2.9), one can think of η as fixed and regard u as a solution to an instance of equation (2.5), where J_s and H_s in (2.5) are determined by η. By condition (iii) in Definition 2.14, this instance of (2.5) corresponds to a nondecreasing homotopy of Hamiltonians. Consequently, the action is nonincreasing along this solution of (2.5) as before. □

It follows from Lemma 2.20 that for any $L \in \mathbb{R}$, we have a subcomplex $CF^{S^1,N,\leqslant L}(H,J)$ of $CF^{S^1,N}(H,J)$, spanned by S^1-orbits of pairs (z,γ) where $z \in \mathrm{Crit}(\tilde{f}_N)$ and γ is a 1-periodic orbit of H_z with $\mathscr{A}_H(z,\gamma) \leqslant L$.

As in §2.2, if the S^1-orbit of (z,γ) is a generator of $CF^{S^1,N}(H,J)$, then there are two possibilities: (i) γ is a constant orbit corresponding to a critical point of H_z on X, and $\mathscr{A}_H(z,\gamma) < \varepsilon$; or (ii) γ is close to a Reeb orbit in $\{\rho\} \times Y$ with period $-h'(e^\rho)$, and $\mathscr{A}_H(z,\gamma)$ is close to this period; in particular $\mathscr{A}_H(z,\gamma) > \varepsilon$.

Definition 2.21. Consider the quotient complex

$$CF^{S^1,N,+}(H,J) := \frac{CF^{S^1,N}(H,J)}{CF^{S^1,N,\leqslant\varepsilon}(H,J)}. \tag{2.14}$$

As in Definition 2.9, the homology of the quotient complex is independent of J, so we can denote this homology by $HF^{S^1,N,+}(H)$; and we have continuation maps $HF^{S^1,N_1,+}(H_1) \to HF^{S^1,N_2,+}(H_2)$ when $(N_1,H_1) \leqslant (N_2,H_2)$. We now define the *positive S^1-equivariant symplectic homology* by

$$SH^{S^1,+}(X,\lambda) := \varinjlim_{N,H} HF^{S^1,N,+}(H). \tag{2.15}$$

Returning to the situation of Remark 2.17, define $HF^{S^1,N,+}(H')$ to be the homology of the quotient of the chain complex (2.11) by the subcomplex spanned by $u^k \otimes \gamma$ where γ is a critical point of H' in X. We then have the following analogue of Proposition 2.18:

Proposition 2.22. The positive S^1-equivariant homology of (X,λ) is given by

$$SH^{S^1,+}(X,\lambda) = \varinjlim_{N\in\mathbb{N},\, H'\in\mathscr{H}_{\text{std}}} HF^{S^1,N,+}(H').$$

2.5. **Example: the ball B^{2n}.**

2.5.1. $SH(B^{2n})$. We consider the ball B^{2n} with the symplectic form which is the restriction of the standard symplectic 2-form $\omega_{std} = \frac{i}{2} dz \wedge d\bar{z} = d\lambda_{std}$ on $\mathbb{C}^n$ and with the Liouville radial vector field defined by $X_{rad} = \frac{1}{2}(z\partial_z + \bar{z}\partial_{\bar{z}})$. The completion is given by $\widehat{B^{2n}} = \mathbb{C}^n$ with the standard symplectic form $\omega_{std} = \frac{i}{2} dz \wedge d\bar{z}$. We look at Hamiltonians

$$H_C : \mathbb{C}^n \to \mathbb{R} : z \mapsto C\,\|z\|^2$$

such that $\frac{C}{\pi} \notin \mathbb{Z}$. These Hamiltonians are not in $\mathscr{H}_{\text{std}}$ but form an admissible cofinite family; see for instance [Oan04, Oan08]. For each C, the Hamiltonian vector field is

$$X_{H_C} = -2iC(z\partial_z - \bar{z}\partial_{\bar{z}}).$$

The integral trajectories are of the form $z(t) = e^{-2iCt} z_0$; therefore, the only 1-periodic orbit of X_H is the critical point $z = 0$. The Floer chain groups are thus

$$CF_*(H_C,J) = \begin{cases} \mathbb{Q} & \text{if } * = -\mathrm{CZ}(0) \\ 0 & \text{otherwise} \end{cases}$$

and, since the differential is 0, the homology groups are the same as the chain groups.

The Conley-Zehnder index of the constant orbit at $z = 0$ depends on C and is given by

$$-\mathrm{CZ}(0) = 2n \left\lfloor \frac{C}{\pi} \right\rfloor + n.$$

Let $C_k := k\pi + \varepsilon$ where $k \in \mathbb{N}_{\geqslant 0}$ and $\varepsilon > 0$. The continuation maps $\varphi_k : CF_*(H_{C_k},J_k) \to CF_*(H_{C_{k+1}},J_{k+1})$ are all identically zero, thus the symplectic homology is also 0

$$SH_*(B^{2n},\lambda_{std}) = \varinjlim_k HF_*(H_{C_k},J_k) = 0.$$

2.5.2. $SH^+(B^{2n})$. We use the long exact sequence :

$$\begin{array}{ccc} H_{*+n}(B^{2n},\partial B^{2n}) & \longrightarrow & SH_*(B^{2n},\lambda_{std}) \\ {\scriptstyle [-1]}\nwarrow & & \swarrow \\ & SH^+_*(B^{2n},\lambda_{std}) & \end{array}$$

and the fact that $SH_*(B^{2n}) = 0$ to deduce that

$$SH^+_*(B^{2n},\lambda_{std}) \cong H_{*+n-1}(B^{2n},S^{2n-1}).$$

To compute the relative homology $H_*(B^{2n},S^{2n-1})$, we use the exact sequence

$$\begin{array}{ccc} H_*(S^{2n-1}) & \longrightarrow & H_*(B^{2n}) \; . \\ {\scriptstyle [-1]}\nwarrow & & \swarrow \\ & H_*(B^{2n},S^{2n-1}) & \end{array}$$

Thus we have

$$H_*(B^{2n},S^{2n-1}) = \begin{cases} \mathbb{Q} & \text{if } * = 2n \\ 0 & \text{otherwise} \end{cases}$$

and this implies

$$SH^+_*(B^{2n},\lambda_{std}) = \begin{cases} \mathbb{Q} & \text{if } * = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular the positive symplectic homology of the ball has only one generator. This shows that this homological invariant cannot detect all distinct periodic Reeb orbits on the sphere (with a non-degenerate contact form).

2.5.3. $SH^{S^1}(B^{2n})$. The idea is the same as in §2.5.1. We take a Hamiltonian

$$H_C : \mathbb{C}^n \times S^{2N+1} \to \mathbb{R} : (z,p) \mapsto C\,\|z\|^2 + \tilde{f}(p)$$

where $\tilde{f}$ is the S^1-invariant lift of a perfect Morse function on $\mathbb{C}P^{n-1}$. The critical points of $\mathscr{A}_H$ are $(p_0,0),\ldots,(p_{N-1},0)$ where $\{p_i \,|\, i = 0\ldots N-1\} = Crit(\tilde{f})$. The index of $(p_i,0)$ is

$$-\operatorname{ind}(0,p_i) + N = 2n\left\lfloor \frac{C}{\pi} \right\rfloor + n + 2i.$$

Therefore

$$CF^{S^1,N}_*(H_C,J) = \begin{cases} \mathbb{Q} & \text{if } * = 2n\left\lfloor \frac{C}{\pi}\right\rfloor + n + 2i, i \in \{0,\ldots N-1\} \\ 0 & \text{otherwise} \end{cases}$$

and, the differential is 0 since the complex is lacunary, thus $HF^{S^1,N}_*(H_C,J) \simeq CF^{S^1,N}_*(H_C)$. If we let $C_k = k\pi(1+2N)+\varepsilon$ for $\varepsilon > 0$, the continuation maps

$\varphi_k : CF_*^{S^1,N}(H_{C_k}, J_k) \to CF_*^{S^1,N}(H_{C_{k+1}}, J_{k+1})$ are identically 0 and thus

$$SH_*^{S^1,N}(B^{2n}, \lambda_{std}) = \varinjlim_N \varinjlim_k HF_*^{S^1,N}(H_{C_k}, J_k) = 0.$$

2.5.4. $SH^{S^1,+}(B^{2n})$. The short exact sequence

$$0 \to CF^{S^1,N,\leqslant\varepsilon}(H,J) \to CF^{S^1,N}(H,J) \to CF^{S^1,N,+}(H,J) \to 0$$

induces a long exact sequence in homology

$$\begin{array}{ccc} H_{*+n}^{S^1}(B^{2n}, \partial B^{2n}) & \longrightarrow & SH_*^{S^1}(B^{2n}, \lambda_{std}) \\ & {}_{[-1]}\nwarrow \quad \swarrow & \\ & SH_*^{S^1,+}(B^{2n}, \lambda_{std}) & \end{array} \tag{2.16}$$

The fact that $SH_*^{S^1,N}(B^{2n}, \lambda_{std}) = 0$ implies

$$SH_*^{S^1,+}(B^{2n}, \lambda_{std}) \cong H_{*+n-1}^{S^1}(B^{2n}, S^{2n-1}).$$

The S^1-action on the pair (B^{2n}, S^{2n-1}) is trivial ; therefore

$$H_*^{S^1}(B^{2n}, S^{2n-1}) = H_*(B^{2n}, S^{2n-1}) \otimes H_*(BS^1).$$

We have as in 2.5.2 that

$$H_*(B^{2n}, S^{2n-1}) = \begin{cases} \mathbb{Q} & \text{if } * = 2n \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$SH_*^{S^1,+}(B^{2n}, \lambda_{std}) = \begin{cases} \mathbb{Q} & \text{if } * = n+1+2i, \quad i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Definition of transfer morphisms

Let (V, λ_V) and (W, λ_W) be Liouville domains. Let $\varphi : V \to W$ be a Liouville embedding, i.e. a smooth embedding such that $\varphi^\star \lambda_W = \lambda_V$. Assume that $\varphi(V) \subset \text{int}(W)$. In this situation one can define a "transfer morphism"

$$\phi_{V,W}^{(S^1,+)} : SH^{(S^1,+)}(W, \lambda_W) \longrightarrow SH^{(S^1,+)}(V, \lambda_V). \tag{3.1}$$

Here the superscript '$(S^1, +)$' means that the superscripts 'S^1' and '$+$' are optional (but the same in all three places).

A transfer morphism for symplectic homology was defined by Viterbo [Vit99], and extended by the author in his PhD thesis [Gut17] for (positive) equivariant symplectic homology. We now review what we need to know about the definition of the transfer morphisms (3.1), and then explain how to extend the construction to generalized Liouville embeddings as in Definition 3.5.

3.1. **Transfer morphisms for (positive) symplectic homology.** To construct transfer morphisms, we introduce a special class $\mathscr{H}_{stair}(V,W)$ of Hamiltonians on $S^1 \times \widehat{W}$ called "admissible stair Hamiltonians". The transfer morphism is defined as a direct limit of continuation morphisms between an admissible Hamiltonian $H_1 \in \mathscr{H}_{\rm std}(W)$ and an admissible stair Hamiltonian $H_2 \in \mathscr{H}_{stair}(V,W)$.

Below, identify V with its image under the Liouville embedding φ. Given $\delta > 0$ small, there is a unique neighbourhood U of ∂V in $W \setminus \mathrm{int}(V)$, together with a symplectomorphism

$$(U,\omega_W) \simeq \big([0,\delta] \times \partial V, d(e^\rho \lambda_V)\big),$$

such that the Liouville vector field for λ_W on the left hand side corresponds to ∂_ρ on the right hand side. Here ρ denotes the $[0,\delta]$ coordinate.

Definition 3.1. A Hamiltonian $H_2 : S^1 \times \widehat{W} \to \mathbb{R}$ is in $\mathscr{H}_{stair}(V,W)$ if and only if

(1): The restriction of H_2 to $S^1 \times V$ is negative, autonomous (i.e. S^1-independent), and C^2-small (so that there are no non-constant 1-periodic orbits). Furthermore,

$$H > -\varepsilon \tag{3.2}$$

on $S^1 \times V$, where $\varepsilon = \frac{1}{2}\min\big\{ \mathrm{Spec}(\partial V, \lambda_V) \cup \mathrm{Spec}(\partial W, \lambda_W)\big\}$.

(2): On $S^1 \times U \cong S^1 \times [0,\delta] \times \partial V$, with ρ denoting the $[0,\delta]$ coordinate, we have:

- There exists $0 < \rho_0 < \frac{\delta}{4}$ such that for $\rho_0 \leqslant \rho \leqslant \delta - \rho_0$ we have

$$H_2(\theta,\rho,y) = \beta e^\rho + \beta', \tag{3.3}$$

  where $y \in \partial V$, $0 < \beta \notin \mathrm{Spec}(\partial V,\lambda_V) \cup \mathrm{Spec}(\partial W,\lambda_W)$ and $\beta' \in \mathbb{R}$.
- There exists a strictly convex increasing function $h_1 : [1, e^{\rho_0}] \to \mathbb{R}$ such that on $S^1 \times [0,\rho_0] \times \partial V$, the function H_2 is C^2-close to the function sending $(\theta,\rho,y) \mapsto h_1(e^\rho)$. Here and in the rest of this definition, the meanings of "close" and "small" are as in Remarks 2.4 and 2.8.
- There exists a small, strictly concave, increasing function $h_2 : [e^{\delta-\rho_0}, e^\delta] \to \mathbb{R}$ such that on $S^1 \times [\delta-\rho_0,\delta] \times \partial V$, the function H_2 is C^2-close to the function sending $(\theta,\rho,y) \mapsto h_2(e^\rho)$.

(3): On $S^1 \times W \setminus (V \cup U)$, the function H_2 is C^2-close to a constant.

(4): On $S^1 \times [0,+\infty) \times \partial W$, with ρ' denoting the $[0,\infty)$ coordinate, we have:

- There exists $\rho_1' > 0$ such that for $\rho' \geqslant \rho_1'$ we have

$$H_2(\theta,\rho',p) = \mu e^{\rho'} + \mu',$$

  with $0 < \mu \notin \mathrm{Spec}(\partial V,\lambda_V) \cup \mathrm{Spec}(\partial W,\lambda_W)$, $\mu < \frac{\beta(e^\delta - 1)}{e^\delta}$, and $\mu' \in \mathbb{R}$.
- There exists a strictly convex, increasing function $h_3 : [1, e^{\rho_1'}] \to \mathbb{R}$ such that $h_3 - h_3(1)$ is small, and on $S^1 \times [0,\rho_1'] \times \partial W$, the function H_2 is C^2-close to the function sending $(\theta,\rho',p) \mapsto h_3(e^{\rho'})$.

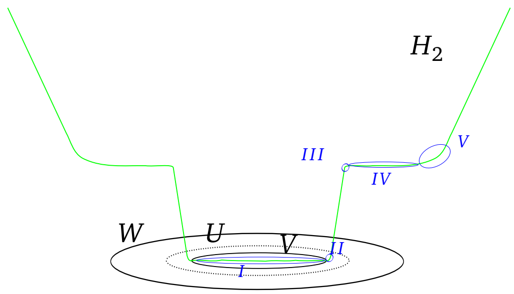

FIGURE 1. [Gut17] Graph of an admissible stair Hamiltonian H_2 on $S^1 \times \widehat{W}$

(5): The Hamiltonian H_2 is nondegenerate, i.e. all 1-periodic orbits of X_{H_2} are nondegenerate.

We denote the set of admissible stair Hamiltonians by $\mathscr{H}_{stair}(V, W)$.

The graph of an admissible stair Hamiltonian H_2 is shown schematically in Figure 1.

The 1-periodic orbits of H_2 lie either in the interior of V (which we call region I), in $[0, \rho_0] \times \partial V$ (region II), in $[\delta - \rho_0, \delta] \times \partial V$ (region III), in $W \backslash (V \cup U)$ (region IV), or in $[0, \rho_1'] \times \partial W$ (region V).

- **I** The 1-periodic orbits in region I correspond to critical points of H_2 on V.
- **II** In region II, the 1-periodic orbits are associated to Reeb orbits of λ_V on ∂V as in Remark 2.8.
- **III** In region III, the 1-periodic orbits are likewise associated to Reeb orbits of λ_V on ∂V.
- **IV** The 1-periodic orbits in region IV correspond to critical points H_2 on $W \backslash (V \cup U)$.
- **V** In region V, the 1-periodic orbits are associated to Reeb orbits of λ_W on ∂W.

The Hamiltonian actions of the 1-periodic orbits are ordered as follows:

$$\mathscr{A}(IV) < \mathscr{A}(V) < 0 < \mathscr{A}(I) < \mathscr{A}(II).$$

This means that every 1-periodic orbit in region IV has Hamiltonian action less than every 1-periodic orbit in region V, and so forth.

We now consider the Floer chain complex $CF(H_2, J_2)$ where $J_2 : S^1 \to \mathrm{End}(T\widehat{W})$ is an S^1-family of almost complex structures on $\widehat{W}$. As in Definition 2.5, we assume that J_2^θ is $\widehat{\omega}_W$-compatible for each $\theta \in S^1$, and that

$$J_2^\theta(\partial_{\rho'}) = R_{\lambda_W}$$

on $[\rho_1', \infty) \times \partial W$. This is enough to give a well-defined chain complex $CF(H_2, J_2)$, cf. [Oan03, §1.2.3]. We also assume that

$$J_2^\theta(\partial_\rho) = R_{\lambda_V} \tag{3.4}$$

on $[\rho_0,\delta-\rho_0]\times\partial V$.

Let $C^{I,III,IV,V}(H_2,J_2)$ denote the subcomplex of $CF(H_2,J_2)$ generated by 1-periodic orbits lying in regions I, III, IV, and V. Let $C^{III,IV,V}(H_2,J_2)$ denote the subcomplex of $CF(H_2,J_2)$ generated by 1-periodic orbits lying in regions III, IV and V. These are subcomplexes because the action decreases along Floer trajectories, and [CO, Lem. 2.3] shows that there does not exist any Floer trajectory from region III to region I or II. We then have quotient chain complexes

$$C^{I,II}(H_2,J_2) = C^{I,II,III,IV,V}(H_2,J_2)/_{C^{III,IV,V}(H_2,J_2)}$$
$$C^{II}(H_2,J_2) = C^{I,II,III,IV,V}(H_2,J_2)/_{C^{I,III,IV,V}(H_2,J_2)}.$$

Given H_2 and J_2 as above, let $H_2^V\in\mathscr{H}_{\mathrm{std}}(V)$ denote the admissible Hamiltonian for V which agrees with H_2 on $V\cup([0,\delta-\rho_0]\times\partial V)$, and which agrees with the right hand side of (3.3) on $[\rho_0,\infty)\times\partial V$. Let J_2^V denote the admissible S^1-family of almost complex structures on $\widehat{V}$ which agrees with J_2 on $V\cup([0,\delta-\rho_0]\times\partial V)$, and which satisfies (3.4) on $[\rho_0,\infty)\times\partial V$. Observe that we have canonical identifications of chain modules

$$\begin{aligned} C^{I,II}(H_2,J_2) &= CF\left(H_2^V,J_2^V\right),\\ C^{II}(H_2,J_2) &= CF^+\left(H_2^V,J_2^V\right), \end{aligned} \tag{3.5}$$

because the generators on both sides correspond to the same 1-periodic orbits in $V\cup([0,\delta-\rho_0]\times\partial V)$.

Proposition 3.2. [Gut17, Proposition 4.4] The canonical identifications (3.5) induce isomorphisms on homology

$$H\big(C^{I,II}(H_2,J_2),\partial\big) = HF\left(H_2^V,J_2^V\right),$$
$$H\big(C^{II}(H_2,J_2),\partial\big) = HF^+\left(H_2^V,J_2^V\right).$$

Given H_2 and J_2 as above, suppose that $H_1\in\mathscr{H}_{\mathrm{std}}(W)$ satisfies $H_1\leqslant H_2$ pointwise. Let J_1 be an admissible S^1-family of almost complex structures on $\widehat{W}$. We then have a well-defined continuation map

$$HF(H_1,J_1)\longrightarrow HF(H_2,J_2) \tag{3.6}$$

defined as in (2.5).

Definition 3.3. We define the transfer morphism on Floer homology to be the composition

$$\phi_{H_2^V,H_1}: HF(H_1,J_1)\longrightarrow HF(H_2,J_2)\longrightarrow H\big(C^{I,II}(H_2,J_2)\big) = HF(H_2^V,J_2^V).$$

Here the first arrow is the continuation map (3.6), the second map is induced by projection onto the quotient chain complex, and the equality sign on the right is the canonical isomorphism from Proposition 3.2. Concretely, this map counts solutions of equation (2.5) going from a 1-periodic orbit of X_{H_1} to a 1-periodic orbit of X_{H_2} lying in region I or II.

Since the continuation map decreases action, it follows that in the above composition, we can start with the homology of the quotient by $CF^{\leqslant\varepsilon}(H_1,J_1)$, to obtain a transfer map on positive Floer homology,

$$\phi^+_{H_2^V,H_1} : HF^+(H_1,J_1) \longrightarrow H\left(\frac{CF(H_2,J_2)}{CF^{\leqslant\varepsilon}(H_2,J_2)}\right) \longrightarrow H\big(C^{II}(H_2,J_2)\big) = HF^+(H_2^V,J_2^V).$$

The above transfer maps $\phi_{H_2^V,H_1}$ and $\phi^+_{H_2^V,H_1}$ depend only on H_1 and H_2^V, and more generally commute with continuation maps for increasing H_1 and H_2^V; see [Gut17, Prop. 4.7]. Consequently, we can define a transfer morphism on (positive) symplectic homology by taking direct limits:

$$\begin{aligned}\phi_{V,W} &= \varinjlim_{H_1,H_2^V} \phi_{H_2^V,H_1} : SH(W,\lambda_W) \longrightarrow SH(V,\lambda_V),\\ \phi^+_{V,W} &= \varinjlim_{H_1,H_2^V} \phi^+_{H_2^V,H_1} : SH^+(W,\lambda_W) \longrightarrow SH^+(V,\lambda_V).\end{aligned}$$

3.2. **Transfer morphisms for (positive) S^1-equivariant symplectic homology.** Recall that to define (positive) S^1-equivariant symplectic homology, we modify the definition of (positive) symplectic homology, by replacing the notion of admissible Hamiltonians $H : S^1 \times \widehat{X} \to \mathbb{R}$ in Definition 2.3 by the notion of admissibile parametrized Hamiltonians $H : S^1 \times \widehat{X} \times S^{2N+1}$ in Definition 2.14. In an analogous way, one can modify the definition of admissible stair Hamiltonians $H_2 : S^1 \times \widehat{W} \to \mathbb{R}$ in Definition 3.1, to define a notion of "admissible parametrized stair Hamiltonians" $H_2 : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$. We can then repeat the constructions in §3.1 to obtain transfer maps

$$\begin{aligned}&\phi^{S^1}_{H_2^V,H_1} : HF^{S^1,N}(H_1) \longrightarrow HF^{S^1,N}(H_2^V),\\ &\phi^{S^1,+}_{H_2^V,H_1} : HF^{S^1,N,+}(H_1) \longrightarrow HF^{S^1,N,+}(H_2^V).\end{aligned} \tag{3.7}$$

We can then take the direct limit over H_1, H_2^V, and N to define transfer morphisms

$$\begin{aligned}\phi^{S^1}_{V,W} &: SH^{S^1}(W,\lambda_W) \longrightarrow SH^{S^1}(V,\lambda_V),\\ \phi^{S^1,+}_{V,W} &: SH^{S^1,+}(W,\lambda_W) \longrightarrow SH^{S^1,+}(V,\lambda_V).\end{aligned}$$

Remark 3.4. One can also describe the transfer morphism (3.7) for fixed N in the context of Remark 2.17 and Proposition 2.22. Here one starts with an admissible stair Hamiltonian $H_2' : S^1 \times \widehat{W} \to \mathbb{R}$ and an admissible Hamiltonian $H_1' : S^1 \times \widehat{V} \to \mathbb{R}$ with $H_1' \leqslant (H_2')^V$. Recall that the homology $HF^{S^1,N,+}(H_1')$ appearing in Proposition 2.22 is the homology of a chain complex generated by symbols $u^k \otimes \gamma$, where $k \in \{0,\ldots,N\}$ and γ is a nonconstant 1-periodic orbit of $X_{H_1'}$. The differential has the form

$$\partial_1^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_{1,i}(\gamma).$$

Likewise, the homology $HF^{S^1,N,+}((H_2')^V)$ is the homology of a chain complex generated by symbols $u^k \otimes \gamma$, where $k \in \{0,\dots,N\}$ and γ is a nonconstant 1-periodic orbit of $X_{(H_2')^V}$. The differential has the form

$$\partial_2^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_{2,i}(\gamma).$$

We now construct the transfer map (3.7) using continuation maps for homotopies which respect the inclusions $\widetilde{\imath}_0$ and $\widetilde{\imath}_1$ as in Remark 2.17. This transfer map will then be induced by a chain map having the form

$$\psi(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \psi_i(\gamma). \tag{3.8}$$

3.3. **Transfer morphisms for generalized Liouville embeddings.** We now extend the definition of transfer morphisms for a generalized Liouville embedding $\varphi : (V,\lambda_V) \to (W,\lambda_W)$ with $\varphi(V) \subset \operatorname{int}(W)$.

Definition 3.5. Let (X,λ) and (X',λ') be Liouville domains of the same dimension. A *generalized Liouville embedding* $(X,\lambda) \to (X',\lambda')$ is a symplectic embedding $\varphi : (X,d\lambda) \to (X',d\lambda')$ such that

$$\left[\,(\varphi^\star\lambda' - \lambda)\big|_{\partial X}\right] = 0 \in H^1(\partial X;\mathbb{R}).$$

Lemma 3.6. Let $\varphi : (V,\lambda_V) \hookrightarrow (W,\lambda_W)$ be a generalized Liouville embedding with $\varphi(V) \subset \operatorname{int}(W)$. Then there exists a 1-form λ_W' on W such that

(1) $d\lambda_W' = d\lambda_W$,
(2) $\lambda_W' = \lambda_W$ near ∂W,
(3) $\varphi^\star\lambda_W' = \lambda_V$.

Proof. Given $\delta > 0$, define

$$V_\delta = V \cup ([0,\delta] \times \partial V) \subset \widehat{V}.$$

As in [MS17, Thm. 3.3.1], if δ is sufficiently small then we can extend φ to a symplectic embedding

$$\varphi_\delta : (V_\delta, \widehat{\omega_V}) \longrightarrow (W,\omega_W).$$

Now use the map φ_δ to identify V_δ with its image in W. Then the 1-form $\lambda_W - \widehat{\lambda_V}$ is closed on V_δ.

By hypothesis, the de Rham cohomology class of this 1-form restricted to $[0,\delta] \times \partial V$ is zero. Thus there is a function $g : [0,\delta] \times \partial V$ such that

$$dg = (\lambda_W - \widehat{\lambda_V})\big|_{[0,\delta]\times\partial V}.$$

Let $\beta : [0,\delta] \to \mathbb{R}$ be a smooth function with $\beta(\rho) \equiv 0$ for ρ close to 0 and $\beta(\rho) \equiv 1$ for ρ close to δ. We can then take

$$\lambda_W' = \begin{cases} \lambda_V & \text{on } V, \\ \widehat{\lambda_V} + d(\beta g) & \text{on } [0,\delta] \times \partial V, \\ \lambda_W & \text{on } W \setminus V_\delta. \end{cases}$$

□

Now given a generalized Liouville embedding as above, let λ'_W be a 1-form on W provided by Lemma 3.6. We then have an honest Liouville embedding

$$\varphi : (V, \lambda_V) \longrightarrow (W, \lambda'_W).$$

As explained in §3.1 and §3.2, this induces transfer maps

$$SH^{(S^1,+)}(W, \lambda'_W) \longrightarrow SH^{(S^1,+)}(V, \lambda_V). \tag{3.9}$$

The construction in §2 of (positive, S^1-equivariant) symplectic homology of (W, λ_W) depends only on the contact form $\lambda_W|_{\partial W}$ on the boundary, and the symplectic form $\omega_W = d\lambda_W$ on the interior. Indeed, replacing the Liouville form λ_W by another Liouville form λ'_W with the same exterior derivative and restriction to the boundary does not change any of the chain complexes or maps in the definition of (positive, S^1-equivariant) symplectic homology[9], since the classes of admissible Hamiltonians used are determined by the restriction to the boundary, and the Hamiltonian vector fields are determined by the symplectic form. (For stronger results on invariance of symplectic homology see §4 or [Gut17, §4.3].) Thus we have a canonical isomorphism

$$SH^{(S^1,+)}(W, \lambda_W) = SH^{(S^1,+)}(W, \lambda'_W). \tag{3.10}$$

We can now finally make the following definition:

Definition 3.7. Suppose $\varphi : (V, \lambda_V) \to (W, \lambda_W)$ is a generalized Liouville embedding with $\varphi(V) \subset \mathrm{int}(W)$. Let λ'_W be a 1-form provided by Lemma 3.6. Define the *transfer morphism*

$$\phi^{(S^1,+)}_{V,W} : SH^{(S^1,+)}(W, \lambda_W) \longrightarrow SH^{(S^1,+)}(V, \lambda_V) \tag{3.11}$$

to be the composition of the canonical isomorphism (3.10) with the map (3.9).

The transfer morphism (3.11) does not depend on the choice of λ'_W, because the admissible Hamiltonians, chain complexes, and chain maps in the definition of the transfer morphism depend only on the symplectic form on each Liouville domain and the contact form on the boundary of each Liouville domain.

4. Invariance of symplectic homology

In this section, we study the invariance of the (S^1-equivariant) positive symplectic homology with respect to the choice of the Liouville vector field in a neighbourhood of the boundary. This has been studied by Viterbo [Vit99], Cieliebak [Cie02] and Seidel [Sei08] in the case of the symplectic homology, in the framework of Liouville domains.

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[9] One might worry that the Hamiltonian action of a noncontractible loop can change if $\lambda_W - \lambda'_W$ is not exact. However for the Hamiltonians that we are using, the only noncontractible 1-periodic orbits are associated to Reeb orbits and their action does not change.

Recall that the completion $(\widehat{X},\widehat{\lambda})$ of the Liouville domain (X,λ) is defined by

$$\widehat{X} := X \cup \big([0,\infty)\times Y\big) \quad \text{and} \quad \widehat{\lambda} := \begin{cases} \lambda & \text{on } X, \\ e^{\rho}\lambda|_Y & \text{on } [0,\infty)\times Y \end{cases}$$

where ρ denotes the $[0,\infty)$ coordinate. The Liouville vector field Z defined by $\iota(Z)d\lambda = \lambda$; it points strictly outwards along Y.

Lemma 4.1. Let (X,λ) be a Liouville domain and let k be a positive real number. Then

$$SH^{\dagger}(X,\lambda) = SH^{\dagger}(X,k\lambda).$$

Where † denotes any of the variants that we have considered $\emptyset, +, S^1$ or $(S^1,+)$.

Proof. The symplectic completions are $(\widehat{X},\widehat{\lambda})$ and $(\widehat{X},k\widehat{\lambda})$; the chain complexes for a pair (H,J) on $(\widehat{X},\widehat{\lambda})$ and the pair (kH,J) on $(\widehat{X},k\widehat{\lambda})$ are the same, since the 1 periodic orbits are the same, and the Floer trajectories satisfy the same equations; indeed $X_H^{\lambda} = X_{kH}^{k\lambda}$. Similarly, continuation maps are equivalent taking as homotopies H_s and kH_s. The result follows, observing that kH form a cofinal family. □

For positive or S^1-equivariant positive homology, we assume that we are in a framework where it is well-defined.

Lemma 4.2. Let (X,λ) and (X',λ') be two Liouville domains. If there exists a symplectomorphism $\varphi : X \to X'$ such that $\varphi(Y) = Y'$, and such that $\varphi_\star(Z) = Z'$ on a neighbourhood of Y then

$$SH^{\dagger}(X,\lambda) \cong SH^{\dagger}(X',\lambda').$$

Proof. We can extend φ to a symplectomorphism $\widehat{\varphi} : \widehat{X} \to \widehat{X}'$ of the completions. For J' an almost complex structure on $\widehat{X}'$, we take the corresponding almost complex structure J on $\widehat{X}$ defined by

$$J_x := \widehat{\varphi}_{\star_x}^{-1} \circ J'_{\widehat{\varphi}(x)} \circ \widehat{\varphi}_{\star_x}$$

and if H' is a Hamiltonian on $\widehat{W'}$, we take the Hamiltonian H on $\widehat{W}$ defined by $H := \widehat{\varphi}^{\star}H'$. Then the 1 periodic orbirs are in bijection and so are the Floer trajectories. The subfamily $\{\widehat{\varphi}^{\star}H'\}$ of Hamiltonians is cofinal and thus we reach the conclusion. □

Lemma 4.3. Let (X,λ) be a Liouville domain. Then for all $R \in \mathbb{R}^+$, we have

$$SH^{\dagger}(X,\lambda) \cong SH^{\dagger}\big(X \cup ([0,R]\times Y),\lambda'\big)$$

where the 1-form λ' on $[0,R]\times\partial X$ is the restriction of the 1-form $\widehat{\lambda}$, thus $(e^{\rho}\lambda_{|_Y})$.

Proof. Denote by φ_t^Z the flow of Z; since $\mathcal{L}_Z\lambda = \lambda$ we have $\varphi_t^{Z\star}\lambda = e^t\lambda$. This gives a symplectomorphism

$$\varphi_R^Z : (X, e^R(d\lambda)) \to \big(X \cup ([0,R]\times Y),\omega'\big)$$

mapping the boundary Y to the boundary $\{R\}\times Y$ and such that ${\varphi_R^Z}^*\lambda = e^R\lambda$. One concludes by the two lemmas above. Explicitely, the diffeomorphism $\varphi_R^Z : \widehat{X}\to\widehat{X}$ maps Hamiltonian vector fields as follows : $(\varphi_R^Z)_*(X_{H'}) = X_H$ when $H' = e^{-R}(\varphi_R^Z)^*H$; hence φ_R^Z gives a bijection between 1-periodic orbits of $X_{H'}$ and 1-periodic orbits of X_H, and, with suitable choices of J's, a bijection between Floer trajectories between 1-periodic orbits of $X_{H'}$ and Floer trajectories between 1-periodic orbits of X_H. Hence it yields an isomorphism

$$HF^\dagger(X, e^{-R}(\varphi_R^Z)^*H) \cong HF^\dagger\big(X\cup([0,R]\times Y), H\big).$$

Furthermore, the diffeomorphism φ_R^Z intertwines a continuation morphism defined by a homotopy H'_s to the corresponding continuation morphism defined by H_s when again $H'_s = e^{-R}(\varphi_R^Z)^*H_s$. This yields the isomorphism mentionned above. □

Lemma 4.4. The transfer morphism

$$SH^\dagger\big(X\cup([0,R]\times Y),\lambda'\big)\mapsto SH^\dagger(X,\lambda)$$

is an isomorphism which coincides with the natural identification of Lemma 4.3.

Proof. Let H be an admissible Hamiltonian for $X\cup([0,R]\times Y)$. Consider the homotopy $H_s^1 := e^{-f(s)}{\varphi_{f(s)}^Z}^\star H$ with $f:\mathbb{R}\to[0,R]$ a smooth function so that $H_s^1 = H$ for large negative s and $H_s^1 = \widetilde{H} := e^{-R}(\varphi_R^Z)^*H$ for large positive s. The set of 1 periodic orbits for H_s^1 is constant (since, as in the Lemma above, the diffeomorphism $\varphi_{f(s)}^Z$ of the completion is a bijection between 1-periodic orbits of $X_{H_s^1}$ and 1-periodic orbits of X_H). This homotopy defines the "transfer morphism"

$$\phi : HF^\dagger\big(X\cup([0,R]\times\partial X), H\big)\to HF^\dagger(X,\widetilde{H}).$$

Let $\{H_s^\eta\}_{\eta\in[0,1]}$ be a family of homotopies (with non fixed endpoint) such that H_s^0 is the constant homotopy $H_s^0 = H$ for all s, and such that all H_s^η are of the form $e^{-f'(s,\eta)}{\varphi_{f'(s,\eta)}^Z}^\star H$ with $f'(.,\eta):\mathbb{R}\to[0,\eta R]$ and $f'(.,1) = f$. We have $H_{+\infty}^\eta = e^{-\eta R}{\varphi_{\eta R}^Z}^\star H = H_{f^{-1}(\eta R)}^1$. The set of 1-periodic orbits of H_s^η is in bijection with the set of orbits of H. We consider, for a given η, the space of Floer trajectories

$$\mathcal{M}(H_s^\eta, J_s^\eta) := \bigcup_{\substack{(\gamma_-^\eta,\gamma_+^\eta)\in\mathcal{P}(H_{-\infty}^\eta)\times\mathcal{P}(H_{+\infty}^\eta)\\ \mathrm{CZ}(\gamma_-^\eta)=\mathrm{CZ}(\gamma_+^\eta)}} \mathcal{M}(\gamma_-^\eta,\gamma_+^\eta,H_s^\eta,J_s^\eta)$$

and the parametrized moduli space

$$\mathcal{M}(\{H_s^\eta,J_s^\eta\}) := \bigcup_{\eta\in[0,1]}\mathcal{M}(H_s^\eta,J_s^\eta)$$

which could have boundaries for some $\eta\neq 0,1$. It defines a cobordism between $\mathcal{M}(H_s^0,J_s^0)$ and $\mathcal{M}(H_s^1,J_s^1)$. Now $\mathcal{M}(H_s^0,J_s^0) = \mathcal{M}(H,J)$ is the space of constant trajectories $\{u(s,\cdot) = \gamma_0(\cdot)\,|\,\gamma_0\in\mathcal{P}(H)\}$. Thus for small η's, say $\eta\leqslant\eta_0$, the cobordism is a bijection, $\mathscr{M}(H_s^\eta, J_s^\eta)$ consists of exactly one Floer trajectory starting from each orbit in $\mathscr{P}(H)$ and arriving at the corresponding orbit in $\mathscr{P}(H_{+\infty}^\eta)$. The morphism induced by $H_s^{\eta_0}$ is thus the natural identification of periodic orbits. Hence the transfer

$$\phi : HF^\dagger\big(X \cup ([0,R] \times Y), H\big) \to HF^\dagger\big(X \cup ([0,R-\varepsilon] \times Y), e^\varepsilon \varphi_\varepsilon^{Z\star} H\big)$$

is the natural identification for $\varepsilon = \eta_0 R$. Now we use the flow of the Liouville vector field, φ_ε^Z, to carry all this construction further and we get the natural identification as the transfer morphism

$$\phi : HF^\dagger\big(X \cup ([0,R-\varepsilon] \times Y), e^\varepsilon \varphi_\varepsilon^{Z\star} H\big) \to HF^\dagger\big(X \cup ([0,R-2\varepsilon] \times Y), e^{2\varepsilon} \varphi_{2\varepsilon}^{Z\ \star} H\big).$$

By induction and functoriality, we get the result. □

Lemma 4.5. Let X be a compact symplectic manifold with contact type boundary. Let λ_t, $t \in [0,1]$ be an isotopy of Liouville forms on X such that in a neighbourhood U of the boundary, $\lambda_t = \lambda_0$. Then

$$SH^\dagger(X, \lambda_0) \cong SH^\dagger(X, \lambda_1).$$

Proof. Remark that we do not require the $d\lambda_t$ to be equal.
Let Z_t be the time dependent vector field defined by

$$\iota(Z_s)(d\lambda_s) = -\big(\tfrac{d}{dt}\lambda(t)_{|_s}\big)$$

and let φ_t be its flow. In the neighbourhood U, the vector field vanishes, $X_s = 0$, and so $\varphi_1^\star \lambda_1 = \lambda_1 = \lambda_0$ on U . Furthermore $\varphi_1^\star d\lambda_1 = d\lambda_0$ because

$$\begin{aligned}
\frac{d}{dt}\varphi_t^\star \lambda_t\Big|_s &= \varphi_s^\star\Big(\frac{d\lambda_t}{dt}\Big|_s\Big) + \varphi_s^\star \mathscr{L}_{Z_s}\lambda_s \\
&= \varphi_s^\star\Big(\frac{d\lambda_t}{dt}\Big|_s\Big) + \varphi_s^\star\big(\iota(Z_s)d\lambda_s + d\iota(Z_s)\lambda_s\big) \\
&= d\Big(\varphi_s^\star\big(\lambda_s(Z_s)\big)\Big).
\end{aligned}$$

This implies that the completions for λ_0 and $\varphi_1^\star \lambda_1$ are the same, therefore, by lemma 4.2,

$$SH^\dagger(X, \lambda_1) = SH^\dagger(X, \varphi_1^\star \lambda_1) = SH^\dagger(X, \lambda_0).$$

□

Theorem 4.6. Let X be a compact symplectic manifold with contact type boundary. Let λ_t, $t \in [0,1]$ be a homotopy of Liouville forms on X. Then

$$SH^\dagger(X, \lambda_0) \cong SH^\dagger(X, \lambda_1).$$

To prove this Proposition, we use the following Proposition from Cieliebak and Eliashberg:

Proposition 4.7 ([CE12], Proposition 11.8). Let X be a compact symplectic manifold with contact type boundary. Let λ_t, $t \in [0,1]$ be a homotopy of Liouville forms on X. Then there exists a diffeomorphism of the completions $f : \widehat{X_0} \to \widehat{X_1}$ such that $f^\star \widehat{\lambda_1} - \widehat{\lambda_0} = dg$ where g is a compactly supported function.

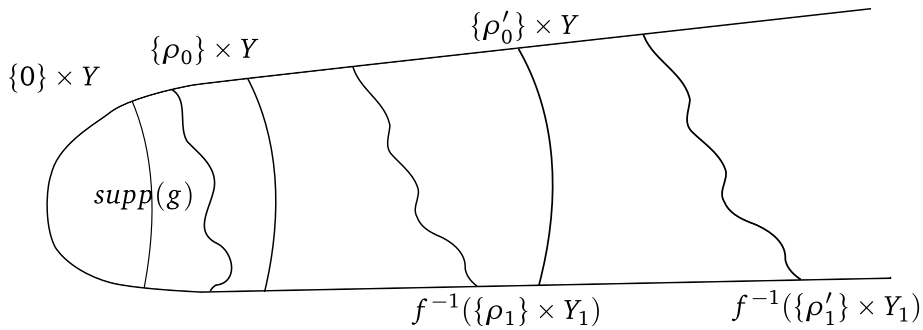


FIGURE 2. The choice of ρ_0, ρ_1, ρ_0' and ρ_1'

Proof of Theorem 4.6. There exists a positive real ρ_0 such that $supp(g) \subset X \cup ([0,\rho_0] \times Y)$. We choose positive real numbers ρ_1, ρ_0' and ρ_1' such that $f^{-1}(X \cup ([0,\rho_1] \times Y))$ contains $X \cup ([0,\rho_0] \times Y)$, $f^{-1}(X \cup ([0,\rho_1] \times Y)) \subset X \cup ([0,\rho_0'] \times Y)$ and $X \cup ([0,\rho_0'] \times Y) \subset f^{-1}(X \cup ([0,\rho_1'] \times Y))$. The situation is represented in Figure 2

The diffeomorphism f and the flow of Z_1 on $\widehat{X}_1$ give

$$\big(f^{-1}(X \cup ([0,\rho_1] \times Y)), f^\star \widehat{\lambda_1}\big) \cong \big(X \cup ([0,\rho_1] \times Y, \widehat{\lambda_1}\big) \cong (X, e^{\rho_1}\lambda_1).$$

The completion of $\big(f^{-1}(X \cup ([0,\rho_1] \times Y)), f^\star \widehat{\lambda_1}\big)$ coincides with $(\widehat{X}_0, \widehat{\lambda_0})$ since close to the boundary $f_\star Z_0 = Z_1$.

$$\begin{aligned} SH(X,\lambda_1) &\cong SH\big(X \cup ([0,\rho_1] \times Y), \widehat{\lambda_1}\big) \quad \text{by Lemma 4.3} \\ &\cong SH\big(f^{-1}(X \cup ([0,\rho_1] \times Y)), f^\star \widehat{\lambda_1}\big) \quad \text{by Lemma 4.2} \\ &\cong SH\big(f^{-1}(X \cup ([0,\rho_1] \times Y)), \widehat{\lambda_0} + dg\big) \quad \text{by Proposition 4.7} \\ &\cong SH\big(\underbrace{f^{-1}(X \cup ([0,\rho_1] \times Y))}_{=:X_1}, \widehat{\lambda_0}\big) \quad \text{by Lemma 4.5.} \end{aligned}$$

Denoting by $\varphi_t^{Z_0}$ the flow of Z_0 and by X_0 the manifold $X \cup ([0,\rho_0] \times Y)$, we have

$$\varphi^{Z_0}_{\rho_1'-\rho_1}(X_1) = f^{-1}(X \cup ([0,\rho_1'] \times Y))$$

and

$$\varphi^{Z_0}_{\rho_0'-\rho_0}(X_0) = X \cup ([0,\rho_0'] \times Y)$$

Using the functoriality of the transfer morphism,

$$\begin{tikzcd} SH(\varphi^{Z_0}_{\rho_1'-\rho_1}(X_1),\widehat{\lambda_0}) \arrow[r] \arrow[rr, bend right, "\cong"'] & SH(\varphi^{Z_0}_{\rho_0'-\rho_0}(X_0),\widehat{\lambda_0}) \arrow[r] \arrow[rr, bend left, "\cong"] & SH(X_1,\widehat{\lambda_0}) \arrow[r] & SH(X_0,\widehat{\lambda_0}); \end{tikzcd}$$

therefore

$$SH(X,\lambda_1) \cong SH(X_1,\widehat{\lambda_0}) \cong SH(X_0,\widehat{\lambda_0}) \cong SH(X,\lambda_0).$$

□

Seidel in [Sei08] has extended the definition of symplectic homology (and all its variants) to Liouville manifolds.

Definition 4.8 (see for instance [CE12]). A *Liouville manifold* is an exact symplectic manifold (W,ω,Z), where the vector field Z is an expanding Liouville vector field, i.e $\mathscr{L}_Z\omega = \omega$ and $\varphi_t^Z\omega = e^t\omega$ such that

- the vector field Z is complete and
- the manifold is convex in the sense that there exists an exhaustion $W = \cup_{k=1}^{\infty} W^k$ by compact domains $W_k \subset W$ with smooth boundaries along which Z is outward pointing.

In the following we will denote a Liouville manifold either by (W,ω,X) or by $(W,\lambda := \iota(Z)\omega)$.

The set

$$\mathrm{Skel}(V,\omega,Z) := \bigcup_{k=1}^{\infty} \bigcap_{t>0} \varphi_{-t}^Z(W^k)$$

is called the *skeleton* of the Liouville manifold (W,ω,Z). It is independent of the choice of the exhausting sequence of compact sets W^k. A Liouville manifold (W,ω,Z) is said to be of *finite type* if its skeleton is compact. Every finite type Liouville manifold is the completion of a Liouville domain[10].

Definition 4.9 ([Sei08]). Let (W,ω,Z) be a Liouville manifold non necessarily of finite type and let W^k be an exhaustion by compact domains $W_k \subset W$ with smooth boundaries along which Z is outward pointing such that $W^k \subset W^{k+1}$. The symplectic homology (and its variants) of (W,λ) is defined as the inverse limit of the symplectic homologies of $(W^k,\lambda_{|_{W^k}})$

$$SH^\dagger(W,\lambda) := \varprojlim SH^\dagger(W^k,\lambda_{|_{W^k}}).$$

The morphisms appearing in this inverse limit are the transfer morphisms.

This definition is independent of the chosen exhaustion. Remark that in the case of finite type Liouville manifolds, this definition coincides with the previous one.

[10]We refer to the book by Cieliebak and Eliashberg for more details, [CE12, Chapter 11]

Proposition 4.10. Let (W_0, λ_0) and (W_1, λ_1) be two Liouville manifolds not necessarily of finite type. Assume there exists an exact symplectomorphism $f : W_0 \to W_1$ i.e. such that $f^\star \lambda_1 - \lambda_0 = dg$ with g a function on W_0. Then

$$SH^\dagger(W_0, \lambda_0) \cong SH^\dagger(W_1, \lambda_1).$$

Proof. Let W_0^k be an exhaustion for W_0 and W_1^k be an exhaustion for W_1 such that for all k,

$$W_0^k \subset f^{-1}(W_1^k) \subset W_0^{k+1}$$

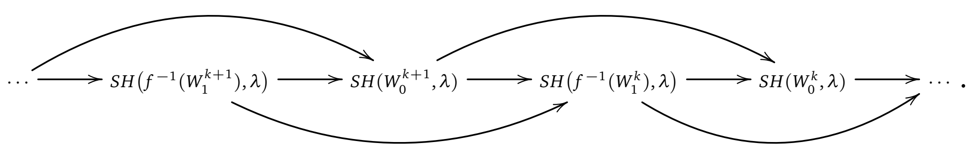
where the inclusion at each level means the inclusion in the interior of the next compact space. Let η be a smooth function $\eta : W_0 \to [0,1]$ such that $\eta = 1$ in a neighbourhood of $\cup_{k=1}^\infty f^{-1}(\partial W_1^k)$ and $\eta = 0$ in a neighbourhood of $\cup_{k=1}^\infty \partial W_0^k$. We define the 1-form λ on W_0 to be

$$\lambda := \lambda_0 + d(\eta g).$$

We have

$$SH(W_0^k, \lambda_0) \cong SH(W_0^k, \lambda) \quad \text{and} \quad SH(W_1^k, \lambda_1) \cong SH\big(f^{-1}(W_1^k), \lambda\big).$$

The functoriality of the transfer morphism implies that the following diagram is commutative:



Therefore,

$$\begin{aligned} SH(W_0, \lambda_0) &\cong \varprojlim SH(W_0^k, \lambda_0) \cong \varprojlim SH(W_0^k, \lambda) \\ &\cong \varprojlim SH\big(f^{-1}(W_1^k), \lambda\big) \cong \varprojlim SH(W_1^k, f_\star \lambda) \\ &\cong \varprojlim SH(W_1^k, \lambda_1) \cong SH(W_1, \lambda_1). \end{aligned}$$

□

The above result may be extended thanks to the following Lemma:

Lemma 4.11 ([BEE12], see also [CE12], Lemma 11.2)**.** Any symplectomorphism between finite type Liouville manifolds $f : (W_0, \lambda_0) \to (W_1, \lambda_1)$ is diffeotopic to an exact symplectomorphism.

We have thus

Theorem 4.12. Let (W_0, λ_0) and (W_1, λ_1) be two Liouville manifolds of finite type such that there exists a symplectomorphism $f : (W_0, \lambda_0) \to (W_1, \lambda_1)$. Then

$$SH^\dagger(W_0, \lambda_0) \cong SH^\dagger(W_1, \lambda_1).$$

4.1. **Invariance of the homology of contact fillings.** In this section we shall prove:

Theorem 4.13. Let (Y_0, ξ_0) and (Y_1, ξ_1) be two contact manifolds that are exactly fillable; i.e. there exist Liouville domains (X_0, λ_0) and (X_1, λ_1) such that $\partial X_0 = Y_0$, $\xi_0 = \ker(\lambda_{0|_{Y_0}})$, $\partial X_1 = Y_1$ and $\xi_1 = \ker(\lambda_{1|_{Y_1}})$. Assume there exists a contactomorphism $\varphi : (Y_0, \xi_0) \to (Y_1, \xi_1)$ which is "oriented" in the sense that $\varphi^\star \lambda_1\big|_{Y_1} = e^f \lambda_0\big|_{Y_0}$. Assume moreover that there exists a contact form $\tilde{\alpha}_0$ on Y_0 such that all periodic Reeb orbits are nondegenerate and the set of all Conley-Zehnder indices is lacunary[11]. Then

$$SH^{S^1,+}(X_0, \lambda_0) \cong SH^{S^1,+}(X_1, \lambda_1).$$

Lemma 4.14 ([Cie02])**.** Let $(\alpha_t)_{t\in[0,1]}$ be a smooth family of contact forms on a closed manifold Y of dimension $2n-1$. Then there exists a $R > 0$ and a non-decreasing function $f : [0,R] \to [0,1]$ such that $f \equiv 0$ close to $\rho = 0$ and $f \equiv 1$ close to $\rho = R$ and

$$d\left(e^\rho \alpha_{f(\rho)}\right) \text{ is symplectic on } [0,R] \times Y.$$

Proof. The proof is a computation:

$$d\left(e^\rho \alpha_{f(\rho)}\right) = e^\rho d\rho \wedge \alpha_{f(\rho)} + e^\rho d\alpha_{f(\rho)} + e^\rho f'(\rho) d\rho \wedge \dot{\alpha}_{f(\rho)}.$$

$$\left(d\left(e^\rho \alpha_{f(\rho)}\right)\right)^n = ne^{n\rho}\left(d\rho \wedge \left(\alpha_{f(\rho)} + f'(\rho)\dot{\alpha}_{f(\rho)}\right) \wedge \left(d\alpha_{f(\rho)}\right)^{n-1}\right)$$

and thus $d\left(e^\rho \alpha_{f(\rho)}\right)$ is symplectic if and only if $\left(\alpha_{f(\rho)} + f'(\rho)\dot{\alpha}_{f(\rho)}\right)\left(R_{\alpha_{f(\rho)}}\right) > 0$. This is true if f' is small. □

Lemma 4.15. If (Y, ξ) is a compact contact manifold which is exactly fillable by a Liouville domain (X, λ_0) (i.e. $\partial X = Y$ and $\xi = \ker \alpha_0$ where $\alpha_0 = \lambda_0\big|_Y$) then, for any contact form α_1 such that $\xi = \ker \alpha_1$ (and α_1 defines the same orientation on Y), there exists a homotopy of Liouville form $\lambda_s, s \in [0,1]$ on X such that $\lambda_1\big|_Y = \alpha_1$.

Proof. Since $\alpha_1 = e^g \alpha_0$, for a smooth function g on Y, we consider the smooth family of contact forms $\alpha_t = e^{tg}\alpha_0, t \in [0,1]$. We define on $X \cup [0,R] \times Y \subset \widehat{X}$ the 1-form $\tilde{\lambda}$:

$$\tilde{\lambda} = \begin{cases} \lambda_0 & \text{on } Y \\ e^\rho \alpha_{f(\rho)} & \text{on } [0,R] \times Y \end{cases}$$

with f as in Lemma 4.14, so that $d\tilde{\lambda}$ is symplectic. The flow of the vector field Z_0, where $\iota(Z_0)d\lambda_0 = \lambda_0$, $\varphi_{-r}^{Z_0}$ induces a diffeomorphism from $X \cup [0,r] \times Y$ to X. The pull-back by this flow of $e^{-r}\tilde{\lambda}$ gives the desired $\lambda_{f(r)}$. □

Combining with Theorem 4.6 and Theorem 5.6, this yields

[11] A set of integer numbers is *lacunary* if it does not contain two consecutive numbers.

Lemma 4.16. Let (Y,ξ) be a contact manifold that is exactly fillable by the Liouville domain (X,λ). Assume that there exists a (oriented) contact form $\tilde{\alpha}$ on Y such that all periodic Reeb orbits are nondegenerate and the set of all Conley-Zehnder indices is lacunary. Then

$$SH^{S^1,+}(X,\lambda)=\bigoplus_{\gamma\in\mathscr{P}(R_{\tilde{\alpha}})}\mathbb{Q}\langle\gamma\rangle$$

where $\mathscr{P}(R_{\tilde{\alpha}})$ denotes the set of periodic Reeb orbits on $(Y,\tilde{\alpha})$.

Proof of Theorem 4.13. Given the contactomorphism $\varphi:(Y_0,\xi_0)\to(Y_1,\xi_1)$ and the contact form $\tilde{\alpha}_0$, we define the form $\tilde{\alpha}_1:=(\varphi^{-1})^\star\tilde{\alpha}_0$; it is a contact form on Y_1 and its periodic orbits are non degenerate, in bijection with those of $\tilde{\alpha}_0$ with the same Conley-Zehnder index. We apply twice Lemma 4.16; once for $(X_0,\lambda_0,\tilde{\alpha}_0)$ and for $(X_1,\lambda_1,\tilde{\alpha}_1)$. □

5. Properties of positive S^1-equivariant SH

Let (X,λ) be a Liouville domain. We now show that the positive S^1-equivariant homology $SH^{S^1,+}(X,\lambda)$ defined in §2, which we denote by $CH(X,\lambda)$ for short, satisfies all of the properties in Proposition 1.6.

5.1. **Free homotopy classes.** Given an admissible Hamiltonan, H, we can decompose the complex $CF^{S^1,N,\leqslant L}(H,J)$ into a direct sum

$$CF^{S^1,N,\leqslant L}(H,J)=\bigoplus_{\Gamma}CF^{S^1,N,\leqslant L}(H,J,\Gamma).$$

Here Γ ranges over free homotopy classes of loops in X, and $CF^{S^1,N,\leqslant L}(H,J,\Gamma)$ denotes the subset of $CF^{S^1,N,\leqslant L}(H,J)$ generated by S^1-orbits of pairs (z,γ) where γ represents the free homotopy class Γ.

The differentials and continuation maps defined in §2 all count certain cylinders, and thus respect the above direct sum decomposition. As a result, we obtain a corresponding direct sum decomposition in (2.14) and (2.15), so that we can decompose

$$CH(X,\lambda)=\bigoplus_{\Gamma}CH(X,\lambda,\Gamma),$$

where $CH(X,\lambda,\Gamma)$ is defined like $CH(X,\lambda)$ but only using loops in the free homotopy class Γ.

Similar remarks apply to all of the constructions to follow; we will omit the free homotopy class Γ below to simplify notation.

5.2. **Action filtration.** Given $L\in\mathbb{R}$, we now define a version of positive S^1-equivariant symplectic homology "filtered up to action L", which we denote by $CH^L(X,\lambda)$. This will only depend on the largest element of $\mathrm{Spec}(Y,\lambda)$ which is less than or equal to L. Thus we can assume without loss of generality that $L\notin\mathrm{Spec}(Y,\lambda)$.

As in Definition 2.21, we can consider the quotient complex

$$CF^{S^1,N,+,\leqslant L}(H,J) := \frac{CF^{S^1,N,\leqslant L}(H,J)}{CF^{S^1,N,\leqslant \varepsilon}(H,J)}.$$

As in Definition 2.9, the homology of the quotient complex is independent of J, so we can denote this homology by $HF^{S^1,N,+,\leqslant L}(H)$. If $(N_1,H_1) \leqslant (N_2,H_2)$, then the continuation chain map induces a well-defined map $HF^{S^1,N_1,+,\leqslant L}(H_1) \to HF^{S^1,N_2,+,\leqslant L}(H_2)$.

Definition 5.1. We define the positive S^1-equivariant symplectic homology filtered up to action L to be

$$CH^L(X,\lambda) := SH^{S^1,+,\leqslant L}(X,\lambda) := \varinjlim_{N,H} HF^{S^1,N,+,\leqslant L}(H).$$

It follows from Remark 2.8(ii) that if $L \notin \operatorname{Spec}(Y,\lambda)$, then $CH^L(X,\lambda)$ depends only on the largest element of $\operatorname{Spec}(Y,\lambda)$ that is less than L.

Given an admissible parametrized Hamiltonian H, a nonnegative integer N, a generic parametrized almost complex structure J as in (2.8), and real numbers $L_1 < L_2$, we have an inclusion of chain complexes

$$CF^{S^1,N,+,\leqslant L_1}(H,J) \longrightarrow CF^{S^1,N,+,\leqslant L_2}(H,J). \tag{5.1}$$

The usual continuation map argument shows that the induced map on homology,

$$HF^{S^1,N,+,\leqslant L_1}(H) \longrightarrow HF^{S^1,N,+,\leqslant L_2}(H), \tag{5.2}$$

does not depend on the choice of J, and commutes with the continuation map for $(N_1,H_1) \leqslant (N_2,J_2)$.

Definition 5.2. We define the map

$$\iota_{L_1,L_2} : CH^{L_1}(X,\lambda) \longrightarrow CH^{L_2}(X,\lambda) \tag{5.3}$$

to be the direct limit over pairs (N,H) of the maps (5.2).

We then have the required property

$$\lim_{L\to\infty} CH^L(X,\lambda) = CH(X,\lambda), \tag{5.4}$$

because we can compute the direct limit

$$\varinjlim_{N,H,L} HF^{S^1,N,+,\leqslant L}(H)$$

either by first taking the limit over pairs (N,H), which gives the left hand side of (5.4), or by first taking the limit over L, which gives the right hand side of (5.4).

Remark 5.3. One can equivalently define $CH^L(X,\lambda)$ by repeating the definition of $CH(X,\lambda)$, but using appropriate admissible Hamiltonians where the limiting slope is equal to L.

5.3. U **map.** We now define the U map on $CH(X,\lambda)$, similarly to [BO16, §2.4].

Recall from Remark 2.17 that given an admissible Hamiltonian $H' : S^1 \times \widehat{X} \to \mathbb{R}$ and a nonnegative integer N, we can choose a pair (H_N, J_N) so that the chain complex $\left(CH^{S^1,N}(H_N,J_N), \partial^{S^1}\right)$ has the nice form given by (2.11) and (2.12).

It follows from (2.12) that the operation of "multiplication by u^{-1}", sending a chain complex generator $u^i \otimes \gamma$ to $u^{i-1} \otimes \gamma$ when $i > 0$ and to 0 when $i = 0$, is a chain map. This induces a map on the homology $HF^{S^1,N}(H')$, which we denote by $U_{N,H'}$. A priori this map also depends on the choice of pair (H_N, J_N), but the usual continuation map argument shows that it does not. In addition, if $(N_1, H_1') \leqslant (N_2, H_2')$, then the continuation map $HF^{S^1,N_1}(H_1') \to HF^{S^1,N_2}(H_2')$ fits into a commutative diagram

$$\begin{array}{ccc} HF^{S^1,N_1}(H_1') & \longrightarrow & HF^{S^1,N_2}(H_2') \\ {\scriptstyle U_{N_1,H_1'}}\downarrow & & \downarrow{\scriptstyle U_{N_2,H_2'}} \\ HF^{S^1,N_1}(H_1') & \longrightarrow & HF^{S^1,N_2}(H_2'). \end{array}$$

It then follows from Proposition 2.18 that we obtain a well-defined map

$$U = \varinjlim_{N,H'} U_{N,H'}$$

on $SH_*^{S^1}(X,\lambda)$.

Since the U map is induced by chain maps which respect (in fact preserve) the symplectic action filtration, it also follows from Proposition 2.22 that we obtain a well-defined U map on $CH(Y,\lambda)$. Similarly we obtain a well-defined U map on $CH^L(Y,\lambda)$. This completes the proof of the "U map" property.

For use in §5.7 below, we also note that there is the following Gysin-type exact sequence:

Proposition 5.4. If (X,λ) is a Liouville domain, then there is a long exact sequence

(5.5)

$$\cdots \longrightarrow SH^+(X,\lambda) \longrightarrow CH(X,\lambda) \xrightarrow{U} CH(X,\lambda) \longrightarrow SH^+(X,\lambda) \longrightarrow \cdots$$

Proof. With the above definition of U, this follows as in [BO16, Prop. 2.9]. This was also shown earlier in [BO13] using a slightly different definition of positive S^1-equivariant symplectic homology. □

5.4. **Reeb Orbits.** Let's start by proving the following Proposition.

Proposition 5.5. If $L_1 < L_2$, and if there does not exist a Reeb orbit γ of $\lambda|_{\partial X}$ in the free homotopy class Γ with action $\mathscr{A}(\gamma) \in (L_1, L_2]$, then the map (1.1) is an isomorphism.

Proof. Let $L_1 < L_2$ such that there does not exist a Reeb orbit γ of $\lambda|_{\partial X}$ having action $\mathscr{A}(\gamma)$ in the interval $(L_1, L_2]$. As in §5.2, we can also assume without loss of generality that $L_1 \notin \mathrm{Spec}(Y,\lambda)$. Then for every triple (N, H, J), if the limiting slope of H is sufficiently large, then the inclusion of chain complexes (5.1) is the identity map. It follows that the map (5.2) is an isomorphism, and consequently the direct limit map (5.3) is an isomorphism as desired. □

We shall leave the proof of the Reeb orbit property (as stated in Proposition 1.6) to the Appendix and prove a preliminary version here. We are in the setup of Remark 2.17.

Theorem 5.6. Let (X,λ) be a Liouville domain. Assume there exists a contact form α on the boundary $\partial X =: Y$ such that the set of Conley-Zehnder indices of all good periodic Reeb orbits is lacunary[12]. Then

$$SH^{S^1,+}(X,\lambda) = \bigoplus_{\gamma\in\mathscr{P}(R_\alpha)} \mathbb{Q}\langle\gamma\rangle$$

where $\mathscr{P}(R_\alpha)$ denotes the set of good periodic Reeb orbits on Y.

Proof. Let H be an admissible Hamiltonian such that the action is distinct for S^1-families of orbits corresponding to Reeb orbits of different period. We consider, as mentioned in Remark 2.17, the S^1-equivariant functions H_N which are lifts of an admissible Hamiltonian $H' \in \mathscr{H}_{\mathrm{std}}$. We use the natural identification, described by (2.11):

$$CF^{S^1,N}(H_N,J_N) = \mathbb{Q}\{1,u,\ldots,u^N\} \otimes_{\mathbb{Q}} CF(H',J_0).$$

and the description of $CF(H',J_0)$ given by Lemma 2.11.

Remark 5.7. The energy $E(u) = \int \|\partial_s u\|^2_{g_{J^\theta_{\eta(s)}}} \, dsd\theta$ of all Floer trajectories involved in the definition of the boundary operator which are linking elements (z,γ) corresponding to distinct γ's, say γ_- and γ_+, is bounded below by some positive constant E depending only on H. Indeed[13], the result follows from the two following facts:
First, $\|\partial_s u\|^2_{g_{J^\theta_{\eta(s)}}}$ is bounded above ([Sal90]) since, otherwise, there would be some "bubbling off" which is prevented by exactness of the symplectic form. Secondly $\int_{S^1} \left\|\partial_\theta u(s,\theta) - X_{H_{\delta,N_{\eta(s)}}}(u(s,\theta))\right\|^2_{g_{J^\theta_{\eta(s)}}} d\theta$ is bounded below by an ε valid for any smooth loop $u(s,\cdot) : S^1 \to \widehat{X} : \theta \mapsto u(s,\theta)$ with some values outside a neighborhood of the critical orbits [Sal99, Exercice 1.22]. This is proven by contradiction, using Arzela-Ascoli Theorem to prove that every sequence $u_n : S^1 \to \widehat{X}$ with $\|\dot{u}_n(t) - X_t(u_n)\|_{L^2} \mapsto 0$ has a subsequence which converges uniformly to a 1-periodic orbit of the Hamiltonian vector field.

The complex $CF^{S^1,N,+}(H_N,J_N)$ is filtered by the action thanks to Proposition **??**. We take the filtration by the action, i.e. by the period ; we define

[12]A set of integer numbers is *lacunary* if it does not contain two consecutive numbers
[13]this argument is borrowed from [GG09]

$F_p CF^{S^1,N,+}(H_N, J_N), p \in \mathbb{Z}$ such that for every $p \in \mathbb{Z}$, the quotient

$$F_{p+1} CF^{S^1,N,+}(H_N, J_N) / F_p CF^{S^1,N,+}(H_N, J_N)$$

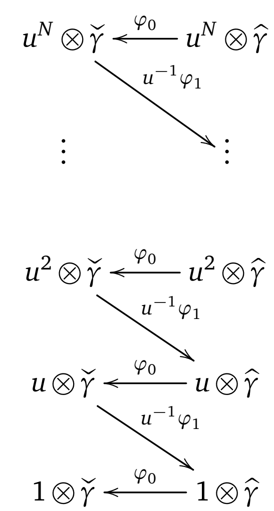
is a union of sets

$$\{1 \otimes \check{\gamma}, \ldots, u^N \otimes \check{\gamma}, 1 \otimes \hat{\gamma}, \ldots, u^N \otimes \hat{\gamma}\}$$

corresponding to underlying Reeb orbits γ of the same period T.
We consider the zero page of the associated spectral sequence.

$$E^{0,N}_{p,q} := F_{p+1} CF^{S^1,N,+}_{p+q}(H_N, J_N) / F_p CF^{S^1,N,+}_{p+q}(H_N, J_N)$$

We have "twin towers of generators", one tower corresponding to each periodic Reeb orbit of period T on $\partial X =: Y$,



with induced differential as in the above diagram with the notation of (2.12).

The differential between two elements in distinct towers of the same period vanishes, since for any Floer trajectory involved in the differential and linking the two towers, by Lemma 2.20 and Remark 5.7,

$$E < \int \|\partial_s u\|^2_{g_{J^\theta_{z(s)}}} ds d\theta \leqslant \mathscr{A}(z^-, \gamma^-) - \mathscr{A}(z^+, \gamma^+)$$

and the last can be chosen to be less than E.

To study any given tower, we use the explicit description of φ_0 and φ_1. It was first described by Bourgeois and Oancea but in their computation, they assumed transversality of contact homology. It was then computed without this assumption by Zhao.

(1) [Zha14, Proposition 6.2], [BO09, Lemma 4.28] Let $\hat{\gamma}$, $\check{\gamma}$ and H' be as above. The moduli space $\mathscr{M}(\hat{\gamma}, \check{\gamma}; H', J)/\mathbb{R}$ consists of two elements; they have opposite signs, due to the choice of a system of coherent

orientations, if and only if the underlying Reeb orbit γ is good. This implies that,

$$\varphi_0(\widehat{\gamma}) \begin{cases} 0 & \text{if } \gamma \text{ is good,} \\ \pm 2\check{\gamma} & \text{if } \gamma \text{ is bad.} \end{cases}$$

Recall that a Reeb orbit is called bad if its Conley-Zehnder index is not of the same parity as the Conley-Zehnder index of the simple Reeb orbit with same image, and an orbit γ_H is bad if the underlying Reeb orbit is bad.

(2) [Zha14, Proposition 6.2], [BO16, Lemma 3.3] With the same notations, the map $\varphi_1 : CF_*^+(H',J) \to CF_{*+1}^+(H',J)$ acts by

$$\varphi_1(\check{\gamma}) = \begin{cases} k_\gamma \widehat{\gamma} & \text{if } \gamma \text{ is good,} \\ 0 & \text{if } \gamma \text{ is bad} \end{cases}$$

where k_γ is the multiplicity of the underlying Reeb orbit γ i.e. γ is a k_γ-fold cover of a simple periodic Reeb orbit.

The complex in $E^{0;N}_{p,q}$ defined by the twin tower corresponding to a good orbit yields

$$\mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{(\times k_\gamma)} \ldots \xrightarrow{(\times k_\gamma)} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{(\times k_\gamma)} \mathbb{Q} \xrightarrow{0} \mathbb{Q}$$

and thus, in the homology $E^{1;N}_{p,q}$, it gives one copy of $\mathbb{Q}$ in degree $-\operatorname{CZ}(\gamma)$ and one copy of $\mathbb{Q}$ in degree $-\operatorname{CZ}(\gamma) + 2N$. The first page is given by

$$E^{1;N} = \bigoplus_{\gamma \in \mathscr{P}(H')} \mathbb{Q}\langle \check{\gamma} \rangle \oplus \mathbb{Q}\langle u^N \otimes \widehat{\gamma} \rangle.$$

There are no bad orbits in the generators of the S^1-equivariant symplectic homology. Indeed the complex in $E^{0;N}_{p,q}$ defined by the twin tower over a bad orbit is :

$$\mathbb{Q} \xrightarrow{\times(\pm 2)} \mathbb{Q} \xrightarrow{0} \ldots \xrightarrow{0} \mathbb{Q} \xrightarrow{\times(\pm 2)} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\times(\pm 2)} \mathbb{Q}$$

and the corresponding homology gives 0 in $E^{1;N}_{p,q}$.

The differential on the first page of the spectral sequence vanishes because of the lacunarity of the set of Conley-Zehnder indices; therefore, for N large enough, it gives the homology :

$$HF^{S^1,N,+}(H_N, J_N) = \bigoplus_{\gamma \in \mathscr{P}(H')} \mathbb{Q}\langle \check{\gamma} \rangle \oplus \mathbb{Q}\langle u^N \otimes \widehat{\gamma} \rangle.$$

The morphism induced by a regular homotopy between two such Hamiltonians (built from standard Hamiltonians close to Morse Bott Hamiltonians) respects the filtration, thanks to Lemma 2.20. We can therefore take the direct limit on the pages over those Hamiltonians which form a cofinal family. The inclusion $S^{2N+1} \hookrightarrow S^{2N+3}$ induces a map

$$E^{1;N} = \bigoplus_{\gamma \in \mathscr{P}(R_\alpha)} \mathbb{Q}\langle \check{\gamma} \rangle \oplus \mathbb{Q}\langle u^N \otimes \widehat{\gamma} \rangle \to E^{1;N+1}.$$

which is the identity on the first factor and zero on the second factor. Taking the direct limit over the inclusion $S^{2N+1} \hookrightarrow S^{2N+3}$ we have

$$SH^{S^1,+}(X,\lambda) = \varinjlim_N E^{1;N} = \bigoplus_{\gamma \in \mathscr{P}(R_\alpha)} \mathbb{Q}\langle \gamma \rangle.$$

□

Remark 5.8. Stricto sensu, in the proof of the above Theorem, we have assumed that the orbits are contractible. Nonetheless Theorem 5.6 is true after extending the definition of $HF^{S^1,+}(H,J)$ to all 1-periodic orbits of H. To deal with non contractible orbits, one chooses for any free homotopy class of loops a, a representative l_a and one chooses a trivialisation of the tangent space along that curve. For the free homotopy class of a contractible loop, l_0 is chosen to be constant loop with constant trivialisation. One ask moreover that $l_{a^{-1}}$ is l_a in the reverse order and with the corresponding trivialisation. The action functional induced by a Hamiltonian H becomes

$$\mathscr{A}(\gamma) := -\int_{[0,1]\times S^1} u^\star \omega - \int_{S^1} H(\theta, \gamma(\theta))d\theta$$

where $u : [0,1] \times S^1 \to X$ is a homotopy from l_a to γ. For any loop γ belonging to the free homotopy class a, one chooses a homotopy $u : [0,1] \times S^1 \to X$ from l_a to γ and one considers the trivialisation of TX on γ induced by u and by the choice of the trivialisation along l_a. Let us observe that any Floer trajectory can only link two orbits in the same free homotopy class and as before, the action decreases along Floer trajectories. As before, the Floer complex is generated by the 1-periodic orbits of H graded by minus their Conley-Zehnder index. The differential "counts" Floer trajectories between two orbits whose difference of grading is 1. The positive version of symplectic homology is defined as before since the set of critical points of H is still a subcomplex : Floer trajectories can only link a critical point to a contractible orbit. All the results stated above extend to this framework.

Corollary 5.9. The only generators that may appear in the positive S^1-equivariant homology are of the form $u^0 \otimes \check{\gamma}$ with $\check{\gamma}$ a good orbit.

Corollary 5.10. The number of good periodic Reeb orbits of periods $\leqslant T$ is bounded below by the rank of the positive S^1-equivariant symplectic homology of action $\leqslant T$.

5.5. δ **map.** To define the delta map, we have the following:

Proposition 5.11. Let (X,λ) be a Liouville domain. Then there is a canonical long exact sequence

$$\begin{array}{ccc} H_*(X,\partial X)\otimes H_*(BS^1) & \longrightarrow & SH^{S^1}(X,\lambda)\ . \\ {}_{\delta}\nwarrow & & \swarrow \\ & SH^{S^1,+}(X,\lambda) & \end{array} \tag{5.6}$$

Proof. For any triple (N,H,J) as in Definition 2.21, by definition we have a short exact sequence of chain complexes

$$0 \longrightarrow CF^{S^1,N,\leqslant\varepsilon}(H,J) \longrightarrow CF^{S^1,N}(H,J) \longrightarrow CF^{S^1,N,+}(H,J) \longrightarrow 0. \tag{5.7}$$

Since continuation maps respect symplectic action, we can take the direct limit of the resulting long exact sequences on homology to obtain a canonical long exact sequence

$$\cdots \longrightarrow SH^{S^1,\leqslant\varepsilon}(X,\lambda) \longrightarrow SH^{S^1}(X,\lambda) \longrightarrow SH^{S^1,+}(X,\lambda) \longrightarrow \cdots \tag{5.8}$$

where we define

$$SH^{S^1,\leqslant\varepsilon}(X,\lambda) = \varinjlim_{N,H} HF^{S^1,N,\leqslant\varepsilon}(H,J). \tag{5.9}$$

To compute (5.9), note that we have a canonical isomorphism

$$HF^{S^1,N,\leqslant\varepsilon}(H,J) = H_*(X,\partial X)\otimes\mathbb{Q}\{1,u,\ldots,u^N\}. \tag{5.10}$$

For proofs of counterparts of this isomorphism for different definitions of S^1-equivariant symplectic homology, see [Vit99, Proposition 1.3] and [BO13, Lemma 4.8]. In our context, the isomorphism (5.10) holds because if we compute the left hand side as in Remark 2.17, then the chain complex comes from the critical points of H' on X, so that we have

$$CF^{S^1,N,\leqslant\varepsilon}(H,J) = C_{\text{Morse}}(X,H')\otimes\mathbb{Q}\{1,u,\ldots,u^N\}. \tag{5.11}$$

Here $C_{\text{Morse}}(X,H')$ denotes the chain complex for the Morse cohomology of H', whose differential counts upward gradient flow lines; and u^i represents the index $2i$ critical point of f_N. The differential on the left side of (5.11) agrees on the right side with the tensor product of the Morse differential and the identity on $\mathbb{Q}\{1,u,\ldots,u^N\}$. Since the gradient of H' points out of X along ∂X, the Morse cohomology agrees with the relative homology $H_*(X,\partial X)$. This proves (5.10), and taking the direct limit over pairs (N,H) gives a canonical isomorphism

$$SH^{S^1,\leqslant\varepsilon}(X,\lambda) = H_*(X,\partial X)\otimes H_*(BS^1). \tag{5.12}$$

Putting this into (5.8) proves the proposition. □

The map δ vanishes on $CH(X,\lambda,\Gamma)$ for every free homotopy class $\Gamma\neq 0$, because the maps in the long exact sequence (5.8) preserve the free homotopy class, and the homology (5.12) is entirely supported in the summand corresponding to $\Gamma=0$.

5.6. **Scaling.** If $(\widehat{X},\widehat{\lambda})$ is the completion of (X,λ), then the completion of $(X,r\lambda)$ is naturally identified with the same manifold $\widehat{X}$, with the 1-form $r\widehat{\lambda}$.

If $H: S^1\times\widehat{X}\to\mathbb{R}$ is an S^1-dependent Hamiltonian, and if X_H denotes the (S^1-dependent) Hamiltonian vector field for H defined using $\widehat{\omega}$, then the Hamiltonian vector field for H defined using $r\widehat{\omega}$ is $r^{-1}X_H$. It follows that if H is an admissible Hamiltonian for (X,λ), then rH is an admissible Hamiltonian for $(X,r\lambda)$, with the same 1-periodic orbits. Note here that $\operatorname{Spec}(Y,r\lambda) = r\operatorname{Spec}(Y,\lambda)$, so the conditions involving the action spectrum are preserved. In particular, if $\varepsilon = \frac{1}{2}\min \operatorname{Spec}(Y,\lambda)$ as usual, then

$$r\varepsilon = \frac{1}{2}\min \operatorname{Spec}(Y, r\lambda).$$

Likewise, if $H : S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$ is an admissible parametrized Hamiltonian for (X,λ), then rH is an admissible parametrized Hamiltonian for $(X, r\lambda)$.

If J is an admissible parametrized almost complex structure (2.8) as needed to define the (positive) S^1-equivariant symplectic homology of (X,λ), then J is not quite admissible for $(X, r\lambda)$, because the condition (2.3) only holds up to a constant. However one can still define (positive) S^1-equivariant symplectic homology using parametrized almost complex structures that satisfy this weaker version of admissibility, cf. [Oan04, §1.3.2], and a continuation argument shows that the resulting (positive) S^1-equivariant symplectic homology will be canonically isomorphic.

Putting this together, we have a canonical isomorphism of chain complexes

$$CF^{S^1,N,\leqslant L}(H,J) = CF^{S^1,N,\leqslant rL}(rH,J).$$

We then have a canonical isomorphism of quotient chain complexes

$$\frac{CF^{S^1,N,\leqslant L}(H,J)}{CF^{S^1,N,\leqslant \varepsilon}(H,J)} = \frac{CF^{S^1,N,\leqslant rL}(rH,J)}{CF^{S^1,N,\leqslant r\varepsilon}(rH,J)}.$$

Taking the direct limit over pairs (N,H) gives the desired canonical isomorphism

$$CH^L(X,\lambda) = CH^{rL}(X, r\lambda).$$

We can also take $L = +\infty$, giving the desired canonical isomorphism

$$CH(X,\lambda) = CH(X, r\lambda).$$

These scaling isomorphisms preserve the U and δ maps, since the U map is purely formal, and the holomorphic curves counted by the δ maps are the same.

5.7. **Star-Shaped Domains.** When X is a nice star-shaped domain, the chain complex $CF^{S^1,N}(H,J)$ has a canonical $\mathbb{Z}$ grading, in which the grading of a pair (z,γ) is $\operatorname{ind}(z) - CZ(\gamma)$. Here $\operatorname{ind}(z)$ denotes the Morse index of the corresponding critical point of f_N, while $CZ(\gamma)$ denotes the Conley-Zehnder index of γ, computed using a global trivialization of TX.

With respect to this grading, the long exact sequence (5.6) has the form

(5.13)
$$\begin{array}{ccc} H_{*+n}(X,\partial X)\otimes H_*(BS^1) & \longrightarrow & SH_*^{S^1}(X) \\ & \nwarrow^{\delta}_{[-1]} \quad \swarrow & \\ & SH_*^{S^1,+}(X) & \end{array}.$$

For a nice star-shaped domain X, we have $SH_*^{S^1}(X) = 0$; see §2.5.3. Assertions (i) and (ii) in the Star-Shaped Domains property follow. (The computation

(1.3) also follows from 5.6 together with the description of the Reeb orbits on the boundary of an irrational ellipsoid.)

To prove assertion (iii), note that for a nice star-shaped domain, the Gysin-type sequence (5.5) with gradings has the form

$$\cdots \longrightarrow SH_k^+(X) \longrightarrow CH_k(X) \xrightarrow{U} CH_{k-2}(X) \longrightarrow SH_{k-1}^+(X) \longrightarrow \cdots .$$

On the other hand, if X is a nice star-shaped domain then

$$SH_*^+(X) = \begin{cases} \mathbb{Q} & \text{if } * = n+1 \\ 0 & \text{otherwise} \end{cases},$$

see §2.5.2. Plugging this with (1.3) in the above Gysin-type sequence, we immediately see that the U map $CH_*(X,\lambda) \to CH_{*-2}(X,\lambda)$ is an isomorphism except when $* = n+1$.

Finally, we need to prove assertion (iv). Suppose that $\lambda_0|_{\partial X}$ is nondegenerate and has no Reeb orbit γ with action $\mathscr{A}(\gamma) \in (L_1, L_2]$ and Conley-Zehnder index $\mathrm{CZ}(\gamma) = n - 1 + 2k$. We need to show that the map

$$\iota_{L_1,L_2} : CH^{L_1}_{n-1+2k}(X,\lambda_0) \longrightarrow CH^{L_2}_{n-1+2k}(X,\lambda_0) \tag{5.14}$$

is surjective. As in §5.2, we can assume without loss of generality that $L_1, L_2 \notin \mathrm{Spec}(Y,\lambda)$.

To prove that (5.14) is surjective, we compute positive S^1-equivariant symplectic homology using an admissible Hamiltonian $H' : S^1 \times \widehat{X} \to \mathbb{R}$ as in Remark 2.17. Furthermore, we assume that H' is perturbed from an admissible Morse-Bott Hamiltonian as in Lemma 2.11, with boundary slope $\beta > L_2$. As a result, if $L < \beta$ is not close to the action of a Reeb orbit, then the chain complex $CF^{S^1,N,+,\leqslant L}(H_N, J_N)$ is generated by symbols $u^k \otimes \check{\gamma}$ and $u^k \otimes \hat{\gamma}$ where $0 \leqslant k \leqslant N$ and γ is a Reeb orbit with action $\mathscr{A}(\gamma) \leqslant L$. Furthermore, the grading of a generator is given by

$$\begin{aligned} |u^k \otimes \check{\gamma}| &= \mathrm{CZ}(\gamma) + 2k, \\ |u^k \otimes \hat{\gamma}| &= \mathrm{CZ}(\gamma) + 2k + 1. \end{aligned}$$

Now fix N, H_N, and J_N. The differential on the chain complex $CF^{S^1,N,+,\leqslant L}(H_N, J_N)$ does not increase the symplectic action of Reeb orbits. This means that we can define an integer-valued filtration $\mathscr{F}$ on the chain complex as follows: Denote the real numbers in the action spectrum $\mathrm{Spec}(Y,\lambda)$ by

$$a_1 < a_2 < \cdots .$$

If γ is a Reeb orbit with action $\mathscr{A}(\gamma) = a_j$, then we define the filtration

$$\mathscr{F}(u^i \otimes \check{\gamma}) = \mathscr{F}(u^i \otimes \hat{\gamma}) = j.$$

Let $\mathscr{F}_j CF^{\leqslant L}$ denote the subcomplex of $CF^{S^1,N,+,\leqslant L}(H_N, J_N)$ spanned by generators with filtration $\leqslant j$. Let

$$\mathscr{G}_j CF^{\leqslant L} = \mathscr{F}_j CF^{\leqslant L} / \mathscr{F}_{j-1} CF^{\leqslant L}$$

denote the associated graded complex.

It is shown in §5.4 and in [Gut17, §3.2] that the homology of $\bigoplus_j \mathscr{G}_j CF^{\leqslant L}$ is generated by $u^0 \otimes \check{\gamma}$ and $u^N \otimes \hat{\gamma}$ where γ ranges over the good Reeb orbits with action less than L. It follows that if N is sufficiently large with respect to k and L, then the degree $n-1+2k$ part of $\bigoplus_j \mathscr{G}_j CF^{\leqslant L}$ is generated by $u^0 \otimes \check{\gamma}$ where γ is a good Reeb orbit with action less than L and Conley-Zehnder index equal to $n-1+2k$. Therefore, the inclusion of chain complexes

$$CF^{S^1,N,+,\leqslant L_1}(H_N, J_N) \longrightarrow CF^{S^1,N,+,\leqslant L_2}(H_N, J_N) \tag{5.15}$$

induces an injection

$$\mathscr{G}_j CF^{\leqslant L_1} \longrightarrow \mathscr{G}_j CF^{\leqslant L_2}$$

for each j. Furthermore, under our assumption on k, L_1, and L_2, if N is sufficiently large, then the above injection in grading $n-1+2k$ is an isomorphism

$$\mathscr{G}_j CF^{\leqslant L_1}_{n-1+2k} \xrightarrow{\simeq} \mathscr{G}_j CF^{\leqslant L_2}_{n-1+2k}$$

for each j. It now follows from the algebraic Lemma 5.12 below that the inclusion (5.15) induces a surjection on the degree $n-1+2k$ homology

$$HF^{S^1,N,+,\leqslant L_1}_{n-1+2k}(H_N, J_N) \longrightarrow HF^{S^1,N,+,\leqslant L_2}_{n-1+2k}(H_N, J_N). \tag{5.16}$$

Lemma 5.12. Let

$$0 = \mathscr{F}_0 C_* \subset \mathscr{F}_1 C_* \subset \cdots \subset \mathscr{F}_J C_* = C_*,$$
$$0 = \mathscr{F}_0 C'_* \subset \mathscr{F}_1 C'_* \subset \cdots \subset \mathscr{F}_J C'_* = C'_*$$

be filtered chain complexes. Denote the associated graded chain complexes by $\mathscr{G}_j C_* = \mathscr{F}_j C_* / \mathscr{F}_{j-1} C_*$ and $\mathscr{G}_j C'_* = \mathscr{F}_j C'_* / \mathscr{F}_{j-1} C'_*$. Let $\phi : C_* \to C'_*$ be a map of filtered chain complexes. For a given grading k, suppose that for each j, the map ϕ induces a surjection $H_k(\mathscr{G}_j C_*) \to H_k(\mathscr{G}_j C'_*)$ and an injection $H_{k-1}(\mathscr{G}_j C_*) \to H_{k-1}(\mathscr{G}_j C'_*)$. Then ϕ induces a surjection $H_k C_* \to H_k C'_*$ and an injection $H_{k-1} C_* \to H_{k-1} C'_*$.

Proof. Since the filtrations are bounded, it is enough to prove by induction on j that ϕ induces a surjection $H_k(\mathscr{F}_j C_*) \to H_k(\mathscr{F}_j C'_*)$ and an injection $H_{k-1}(\mathscr{F}_j C_*) \to H_{k-1}(\mathscr{F}_j C'_*)$. Assume that the claim holds for $j-1$. We then have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H_k(\mathscr{F}_{j-1} C_*) & \longrightarrow & H_k(\mathscr{F}_j C_*) & \longrightarrow & H_k(\mathscr{G}_j C_*) & \longrightarrow & H_{k-1}(\mathscr{F}_{j-1} C_*) \\
\downarrow \text{surj} & & \downarrow & & \downarrow \text{surj} & & \downarrow \text{inj} \\
H_k(\mathscr{F}_{j-1} C'_*) & \longrightarrow & H_k(\mathscr{F}_j C'_*) & \longrightarrow & H_k(\mathscr{G}_j C'_*) & \longrightarrow & H_{k-1}(\mathscr{F}_{j-1} C'_*)
\end{array}$$

where the vertical arrows are induced by ϕ. Surjectivity of the second vertical arrow then follows from chasing this diagram. (This is one of the two "four-lemmas" that imply the "five lemma".) Likewise, the injectivity claim for

j follows by chasing the commutative diagram with exact rows

$$\begin{array}{ccccccc} H_k(\mathscr{G}_j C_*) & \longrightarrow & H_{k-1}(\mathscr{F}_{j-1} C_*) & \longrightarrow & H_{k-1}(\mathscr{F}_j C_*) & \longrightarrow & H_{k-1}(\mathscr{G}_j C_*) \\ \downarrow {\scriptstyle \text{surj}} & & \downarrow {\scriptstyle \text{inj}} & & \downarrow & & \downarrow {\scriptstyle \text{inj}} \\ H_k(\mathscr{G}_j C'_*) & \longrightarrow & H_{k-1}(\mathscr{F}_{j-1} C'_*) & \longrightarrow & H_{k-1}(\mathscr{F}_j C'_*) & \longrightarrow & H_{k-1}(\mathscr{G}_j C'_*). \end{array}$$

Since (5.16) is a surjection, by taking the direct limit over N and H', and using an action-filtered version of Proposition 2.22, we conclude that the map (5.14) is surjective as desired. □

6. Properties of transfer maps

Let $\varphi : (X, \lambda) \to (X', \lambda')$ be a generalized Liouville embedding with $\varphi(X) \subset \operatorname{int}(X')$. Let

$$\Phi : CH(X', \lambda') \longrightarrow CH(X, \lambda)$$

denote the transfer map $\phi^{S^1,+}_{X,X'}$ defined in §3. We now prove that this map satisfies the properties in Proposition 1.7.

6.1. **Action.** The transfer map Φ is a direct limit over H_1, H_2^X, and N of continuation maps

$$HF^{S^1,N,+}(H_1) \longrightarrow HF^{S^1,N,+}(H_2^X) \tag{6.1}$$

where H_1 and H_2^X are appropriate parametrized Hamiltonians for X' and X respectively. Since the continuation map (6.1) is induced by a chain map which decreases symplectic action, it is the direct limit over L of maps

$$HF^{S^1,N,+,\leqslant L}(H_1) \longrightarrow HF^{S^1,N,+,\leqslant L}(H_2^X). \tag{6.2}$$

We now define

$$\Phi^L : CH^L(X', \lambda') \longrightarrow CH^L(X, \lambda)$$

to be the direct limit over H_1, H_2^X, and N of the maps (6.2). Here, as in §5.2, we assume without loss of generality that $L \notin \operatorname{Spec}(\partial X', \lambda') \cup \operatorname{Spec}(\partial X, \lambda)$. The required properties (1.4) and (1.5) follow from Definition 5.2.

6.2. **Functoriality.** The proof results from the comparison of a count of Floer trajectories. On one hand, one counts Floer trajectories corresponding to an increasing homotopy H_{13}, going from a 1-periodic orbit of X_{H_1} for an admissible Hamiltonian H_1 on $S^1 \times \widehat{V}_3$ to the $C^{II,I}$ part of a stair Hamiltonian H_3 with two "steps". On the other hand, one counts trajectories relative to the composition of two increasing homotopies, H_{12} going from H_1 to H_2 (a stair Hamiltonian with one step) and H_{23} going from H_2 to H_3. The property is a consequence of the composition of homotopies.

6.3. **Commutativity with U.** We now show that the transfer map Φ commutes with the U map defined in §5.3.

Recall that the map Φ can be computed as a direct limit of maps (3.8) from Remark 3.4. And recall from §5.3 that in this setup, the U map is the direct limit of chain maps given by "multiplication by u^{-1}". So it is enough to prove that for each nonnegative integer N, we have a commutative diagram of chain maps

$$\begin{array}{ccc} CF^{S^1,N,+}(H_1') & \xrightarrow{\psi} & CF^{S^1,N,+}\big((H_2')^V\big) \\ {\scriptstyle u^{-1}}\downarrow & & \downarrow{\scriptstyle u^{-1}} \\ CF^{S^1,N,+}(H_1') & \xrightarrow{\psi} & CF^{S^1,N,+}\big((H_2')^V\big). \end{array}$$

Here the chain complexes depend on S^{2N+1}-families of Hamiltonians and almost complex structures as in Remark 2.17, which we are omitting from the notation.

It is enough to check this commutativity on a generator $u^k \otimes \gamma$. If $k = 0$, then both compositions are zero, since ψ does not increase the exponent of u. If $k > 0$, then the lower left composition is given by

$$\psi\big(u^{-1}(u^k \otimes \gamma)\big) = \psi(u^{k-1} \otimes \gamma) = \sum_{i=0}^{k-1} u^{k-1-i} \otimes \psi_i(\gamma),$$

while the upper right composition is given by

$$u^{-1}\psi(u^k \otimes \gamma) = u^{-1} \sum_{i=0}^{k} u^{k-i} \otimes \psi_i(\gamma) = \sum_{i=0}^{k-1} u^{k-i-1} \otimes \psi_i(\gamma).$$

These are equal, and this completes the proof that $\Phi U = U \Phi$.

To prove that $\Phi^L U^L = U^L \Phi^L$, as before we can assume without loss of generality that $L \notin \operatorname{Spec}(\partial X', \lambda') \cup \operatorname{Spec}(\partial X, \lambda)$. We then repeat the above argument, restricted to orbits with action less than L.

6.4. **Commutativity with δ.** To conclude, we now prove the commutativity with δ in Proposition 1.7. Note that a closely related result was proved in [Vit99, Thm. 5.2], and our proof will use some of the same ideas.

Recall that the δ map is defined starting from the short exact sequence of chain complexes (5.7). If H_1 and H_2^X are Hamiltonians as in the definition of the transfer map in §3.2, then we have a commutative diagram

$$\begin{array}{ccccc} C_{\mathrm{Morse}}(X', H_1) \otimes \mathbb{Q}\{1, u, \ldots, u^N\} & \longrightarrow & CF^{S^1,N}(H_1, J_1) & \longrightarrow & CF^{S^1,N,+}(H_1, J_1) \\ \downarrow & & \downarrow & & \downarrow \\ C_{\mathrm{Morse}}(X, H_2^X) \otimes \mathbb{Q}\{1, u, \ldots, u^N\} & \longrightarrow & CF^{S^1,N}(H_2^X, J_2^X) & \longrightarrow & CF^{S^1,N,+}(H_2^X, J_2^X). \end{array}$$

Here the rows are from the short exact sequences of chain complexes (5.7) for X' and X. The center vertical arrow is the continuation chain map which, in the direct limit, gives the transfer morphism $\phi_{X,X'}^{S^1}$. The right vertical arrow is the continuation chain map which, in the direct limit, gives the transfer morphism

$\Phi = \phi^{S^1,+}_{X,X'}$. The left vertical arrow is the restriction of the center vertical arrow. As in the proof of [Vit99, Thm. 5.2], this left arrow simply discards critical points in $X'\backslash X$ (here we are identifying X with its image in X' under the symplectic embedding), and is the Morse continuation map from $H_1|_X$ to $H_2|_X$.

The above commutative diagram gives rise to a morphism of long exact sequences on homology. One square of this is the commutative diagram

$$\begin{array}{ccc} HF^{S^1,N,+}(H_1,J_1) & \longrightarrow & H_*(X',\partial X')\otimes \mathbb{Q}\{1,u,\dots,u^N\} \\ \phi^{S^1,+}_{H_2^X,H_1}\big\downarrow & & \big\downarrow \rho\otimes 1 \\ HF^{S^1,N,+}(H_2^X,J_2^X) & \longrightarrow & H_*(X,\partial X)\otimes \mathbb{Q}\{1,u,\dots,u^N\}. \end{array}$$

Here the horizontal arrows are the connecting homomorphisms which, in the direct limit, give the δ maps for X' and X. Thus taking the direct limit over N, H_1, and H_2^X, we obtain the desired commutative diagram (1.8).

Appendix A. Proof of the Reeb orbits property

The purpose of this appendix is to prove the Reeb orbits property. The main ingredient is an algebraic lemma concerning filtered complexes which shows that, up to isomorphism, the images of inclusion-induced maps between the filtered parts of the complexes can be recovered from the filtered homology of a new chain complex whose underlying vector space is the E^1 term of the spectral sequence associated to the original filtered complex. This lemma is proven in the following section, and in the subsequent section we apply this together with results from [Gut17],[GH17] to complete the proof of the Reeb orbits property.

A.1. **A lemma on filtered complexes.**

In this section we consider a $\mathbb{Z}$-graded chain complex (C_*,∂) of vector spaces over a field K equipped with a filtration

$$\{0\} = F_0C_* \subset F_1C_* \subset \cdots \subset F_rC_* \subset \cdots \subset C_*$$

(where each F_rC_* is a subcomplex of C_*) that is bounded below by zero and exhausting (*i.e.* $F_\infty C_* := \cup_r F_rC_*$ is equal to C_*). We extend the above filtration by $\mathbb{N}$ to a filtration by $\mathbb{Z}$ by setting $F_iC_* = \{0\}$ for $i<0$.

Recall that the associated graded complex of (C_*,∂), denoted $\mathscr{G}(C_*)$ is the direct sum of quotient complexes $\bigoplus_{p\geqslant 1}\frac{F_pC_*}{F_{p-1}}$, equipped with obvious boundary operator induced from ∂. The homology $H_*\big(\mathscr{G}(C_*)\big)$ evidently splits as a direct sum

$$H_k\big(\mathscr{G}(C_*)\big) = \bigoplus_{p\geqslant 1} H_k\left(\frac{F_pC_*}{F_{p-1}C_*}\right).$$

The following is the main algebraic input needed:

Lemma A.1. With notation and assumptions as above, there is a chain complex (D_*,δ) equipped with a filtration

$$\{0\} = F_0D_* \subset F_1D_* \subset \cdots \subset F_rD_* \subset \cdots \subset D_*$$

where for each r, k

$$F_r D_k = \bigoplus_{1\leqslant p\leqslant r} H_k\left(\frac{F_p C_*}{F_{p-1}C_*}\right) \tag{A.1}$$

and $F_\infty D_* := \cup_r F_r D_* = D_*$, such that the boundary operator δ on D_* strictly lowers filtration in the sense that $\delta(F_r D_*) \subset F_{r-1}D_*$, and such that for $1 \leqslant s \leqslant t \leqslant \infty$ there exists an isomorphism of vector spaces

$$\operatorname{Im}\left(H_k(F_s C_*, \partial) \to H_k(F_t C_*, \partial)\right) \cong \operatorname{Im}\left(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)\right)$$

where the maps on both sides are induced by inclusion of filtered subcomplexes.

The proof of Lemma A.1 will occupy the rest of this section. To begin, let us recall from [Wei94, Section 5.4] some ingredients in the construction of the spectral sequence associated to the filtration on (C_*, ∂).

For $p \in \mathbb{Z}$ write $\eta_p\colon F_p C_* \to \frac{F_p C_*}{F_{p-1}C_*}$ for the natural projection, and for $p, q, r \in \mathbb{Z}$ define:

$$A^r_{p,q} = \left\{x \in F_p C_{p+q} | \partial x \in F_{p-r}C_{p+q-1}\right\},$$
$$\hat{Z}^r_{p,q} = \eta_p(A^r_{p,q}), \qquad \hat{B}^r_{p,q} = \eta_p\left(\partial(A^{r-1}_{p+r-1,q-r+2})\right).$$

For any $r \geqslant 1$ one then has inclusions

$$\{0\} = \hat{B}^0_{p,q} \subset \hat{B}^1_{p,q} \subset \cdots \subset \hat{B}^r_{p,q} \subset \hat{B}^{r+1}_{p,q} \subset \hat{Z}^{r+1}_{p,q} \subset \hat{Z}^r_{p,q} \subset \cdots \subset \hat{Z}^0_{p,q} = \frac{F_p C_{p+q}}{F_{p-1}C_{p+q}}.$$

We also write

$$\hat{B}^\infty_{p,q} = \cup_{r=1}^\infty \hat{B}^r_{p,q} = \cup_{r=1}^\infty \eta_p\left(\partial(A^{r-1}_{p+r-1,q-r+2})\right).$$

Note also that since we assume that $F_i C_* = \{0\}$ for $i \leqslant 0$, we have

$$\hat{Z}^r_{p,q} = \hat{Z}^p_{p,q} = \eta_p\left(\ker \partial|_{F_p C_{p+q}}\right) \text{ for } r \geqslant p.$$

Accordingly if we let $\hat{Z}^\infty_{p,q} = \hat{Z}^p_{p,q}$ then we will have

$$\hat{Z}^\infty_{p,q} = \cap_{r=1}^\infty \hat{Z}^r_{p,q}.$$

As is standard, we write

$$E^r_{p,q} = \frac{\hat{Z}^r_{p,q}}{\hat{B}^r_{p,q}}$$

for $r \in \mathbb{N} \cup \{\infty\}$. For the case that $r = 1$, notice that $\hat{Z}^1_{p,q}$ is equal to the set of degree-$(p+q)$ cycles in the quotient complex $\frac{F_p C_*}{F_{p-1}C_*}$ and that $\hat{B}^1_{p,q}$ is equal to the set of degree-$(p+q)$ boundaries in $\frac{F_p C_*}{F_{p-1}C_*}$; thus

$$E^1_{p,q} = H_{p+q}\left(\frac{F_p C_*}{F_{p-1}C_*}\right). \tag{A.2}$$

The following is standard and easily-checked:

Proposition A.2. (cf. [Wei94, Construction 5.4.6]) For each p,q,r, the boundary operator ∂ induces a map

$$\hat{\partial}^r_{p,q}\colon E^r_{p,q}\to E^r_{p-r,q+r-1}$$

such that

$$\ker(\hat{\partial}^r_{p,q})=\pi(\hat{Z}^{r+1}_{p,q})\quad\text{and}\quad \mathrm{Im}(\hat{\partial}^r_{p+r,q-r+1})=\pi(\hat{B}^{r+1}_{p,q}),$$

where $\pi\colon \hat{Z}^r_{p,q}\to\frac{\hat{Z}^r_{p,q}}{\hat{B}^r_{p,q}}$ is the quotient projection.

We also have the following fact concerning the maps $H_k(F_pC_*,\partial)\to H_k(F_tC_*,\partial)$ for $p\leqslant t$ induced by inclusion of filtered subcomplexes; this is a slight extension of the familiar fact that the spectral sequence of a suitable filtered complex converges to the associated graded of the homology.

Proposition A.3. Let $1\leqslant p\leqslant t\leqslant\infty$ with $p<\infty$. Then there is an isomorphism

$$\frac{\mathrm{Im}\left(H_k(F_pC_*,\partial)\to H_k(F_tC_*,\partial)\right)}{\mathrm{Im}\left(H_k(F_{p-1}C_*,\partial)\to H_k(F_tC_*,\partial)\right)}\cong\frac{\hat{Z}^{\infty}_{p,k-p}}{\hat{B}^{t-p+1}_{p,k-p}}.$$

(Here for the case $t=\infty$ we interpret $F_\infty C_*$ as C_* and $\hat{B}^{\infty-p+1}_{p,k-p}$ as $\hat{B}^{\infty}_{p,k-p}$.)

Proof. There is an obvious surjective map

$$\phi\colon\ \ker(\partial|_{F_pC_k})\to\frac{\mathrm{Im}\left(H_k(F_pC_*,\partial)\to H_k(F_tC_*,\partial)\right)}{\mathrm{Im}\left(H_k(F_{p-1}C_*,\partial)\to H_k(F_tC_*,\partial)\right)}$$

given by including $\ker(\partial|_{F_pC_k})$ into $\ker(\partial|_{F_tC_k})$, then taking homology classes, and then projecting. We see that $x\in\ker(\phi)$ if and only if there is $y\in\ker(\partial|_{F_{p-1}C_k})$ such that x and y represent the same homology class in $H_k(F_tC_*,\partial)$; this holds if and only if we can write $x=y+\partial z$ with $z\in F_tC_{k+1}$, and in this case we would have $z\in A^{t-p}_{t,k-t+1}$ since $\partial z=x-y\in F_pC_k$. Thus $\ker(\phi)=\ker(\partial|_{F_{p-1}C_k})+\partial(A^{t-p}_{t,k-t+1})$ and hence

$$\frac{\mathrm{Im}\left(H_k(F_pC_*,\partial)\to H_k(F_tC_*,\partial)\right)}{\mathrm{Im}\left(H_k(F_{p-1}C_*,\partial)\to H_k(F_tC_*,\partial)\right)}\cong\frac{\ker(\partial|_{F_pC_k})}{\ker(\partial|_{F_{p-1}C_k})+\partial(A^{t-p}_{t,k-t+1})}. \tag{A.3}$$

(The above discussion implicitly assumes that $t<\infty$, but since $\cup_{s=1}^{\infty}F_sC_*=C_*$ the reasoning is equally valid for $t=\infty$ provided that we interpret the notation $A^{t-p}_{\infty,k-\infty+1}$ as $\cup_{p\leqslant s\in\mathbb{N}}A^{s-p}_{s,k-s+1}$, as we will continue to do below).

On the other hand the projection $\eta_p\colon F_pC_k\to\frac{F_pC_k}{F_{p-1}C_k}$ sends $\ker(\partial|_{F_pC_k})$ to $\hat{Z}^{\infty}_{p,k-p}$ and sends $\ker(\partial|_{F_{p-1}C_k})+\partial(A^{t-p}_{t,k-t+1})$ to $\hat{B}^{t-p+1}_{p,k-p}$, and it is easy to check that the resulting map

$$\eta\colon\ \frac{\ker(\partial|_{F_pC_k})}{\ker(\partial|_{F_{p-1}C_k})+\partial(A^{t-p}_{t,k-t+1})}\to\frac{\hat{Z}^{\infty}_{p,k-p}}{\hat{B}^{t-p+1}_{p,k-p}}$$

is an isomorphism. Combining this isomorphism with (A.3) proves the proposition. □

For $1 \leqslant r \leqslant \infty$ let

$$B^r_{p,q} = \frac{\hat{B}^r_{p,q}}{\hat{B}^1_{p,q}} \qquad Z^r_{p,q} = \frac{\hat{Z}^r_{p,q}}{\hat{B}^1_{p,q}},$$

so for $r < p$ we have a chain of inclusions

$$\{0\} = B^1_{p,q} \subset \cdots \subset B^r_{p,q} \subset B^{r+1}_{p,q} \subset \cdots \subset B^\infty_{p,q} \subset Z^\infty_{p,q} = Z^p_{p,q} \subset Z^{r+1}_{p,q} \subset Z^r_{p,q} \subset \cdots \subset Z^1_{p,q} = E^1_{p,q}.$$

Projecting away $\hat{B}^1_{p,q}$ induces isomorphisms $E^r_{p,q} \cong \frac{Z^r_{p,q}}{B^r_{p,q}}$. For each $p, q \in \mathbb{Z}$ and $r \geqslant 1$ let us choose:

- A complement $H^r_{p,q}$ to the subspace $B^r_{p,q}$ within the vector space $Z^r_{p,q}$, and
- A complement $M^r_{p,q}$ to the subspace $Z^{r+1}_{p,q}$ within the vector space $Z^r_{p,q}$.

Given these choices, the projection $Z^r_{p,q} \to E^r_{p,q}$ restricts to $H^r_{p,q}$ as an isomorphism, so the maps $\hat{\partial}^r_{p,q}$ from Proposition A.2 induce maps

$$\partial^r_{p,q} \colon H^r_{p,q} \to H^r_{p-r,q+r-1}$$

with

$$\ker \partial^r_{p,q} = Z^{r+1}_{p,q} \cap H^r_{p,q}, \qquad \operatorname{Im} \partial^r_{p+r,q-r+1} = B^{r+1}_{p,q} \cap H^r_{p,q}.$$

(In particular, since $Z^{r+1}_{p,q} = Z^r_{p,q}$ for $r \geqslant p$, we have $\partial^r_{p,q} = 0$ for $r \geqslant p$).

For any $r \geqslant 2$ the various direct sum decompositions $Z^{j-1}_{p,q} = Z^j_{p,q} \oplus M^{j-1}_{p,q}$ yield a direct sum decomposition

$$\begin{aligned} E^1_{p,q} = Z^1_{p,q} &= Z^r_{p,q} \oplus M^{r-1}_{p,q} \oplus \cdots \oplus M^1_{p,q} \\ &= H^r_{p,q} \oplus B^r_{p,q} \oplus M^{r-1}_{p,q} \oplus \cdots \oplus M^1_{p,q}. \end{aligned}$$

(For $r = 1$ we have $B^1_{p,q} = \{0\}$ and $H^1_{p,q} = E^1_{p,q}$ and the above direct sum decomposition degenerates to $E^1_{p,q} = H^1_{p,q}$).

We accordingly extend our map $\partial^r_{p,q} \colon H^r_{p,q} \to H^r_{p,q}$ to a linear map (still denoted $\partial^r_{p,q}$) defined on all of $E^1_{p,q}$ by setting it equal to zero on the summands $B^r_{p,q}, M^{r-1}_{p,q}, \ldots, M^1_{p,q}$. We also regard the codomain of $\partial^r_{p,q}$ as $E^1_{p,q}$ rather than the subspace $H^r_{p,q}$. With this extended definition, we have

$$\ker \partial^r_{p,q} = (Z^{r+1}_{p,q} \cap H^r_{p,q}) \oplus B^r_{p,q} \oplus M^{r-1}_{p,q} \oplus \cdots \oplus M^1_{p,q} = Z^{r+1}_{p,q} \oplus M^{r-1}_{p,q} \oplus \cdots \oplus M^1_{p,q},$$

where we have used that $B^r_{p,q} \subset Z^{r+1}_{p,q} \subset Z^r_{p,q} = H^r_{p,q} \oplus B^r_{p,q}$, so that $(Z^{r+1}_{p,q} \cap H^r_{p,q}) \oplus B^r_{p,q} = Z^{r+1}_{p,q}$. Since we have a direct sum decomposition

$$E^1_{p,q} = Z^{r+1}_{p,q} \oplus M^r_{p,q} \oplus M^{r-1}_{p,q} \cdots \oplus M^1_{p,q},$$

it follows that:

Corollary A.4. The maps $\partial^r_{p,q}\colon E^1_{p,q} \to E^1_{p-r,q+r-1}$ restrict as isomorphisms $M^r_{p,q} \to B^{r+1}_{p-r,q+r-1} \cap H^r_{p-r,q+r-1}$, and vanish identically on the complementary subspace $Z^{r+1}_{p,q} \oplus M^{r-1}_{p,q} \oplus \cdots \oplus M^1_{p,q}$ to $M^r_{p,q}$ in $E^1_{p,q}$.

In particular, since for $j > r$ we have $M^j_{p,q} \subset Z^j_{p,q} \subset Z^{r+1}_{p,q} \subset \ker(\partial^r_{p,q})$, this shows that $\partial^r_{p,q}$ vanishes on $M^j_{p,q}$ for $j \neq r$, while it maps $M^r_{p,q}$ isomorphically to $B^{r+1}_{p-r,q+r-1} \cap H^r_{p-r,q+r-1}$.

Now for any p, q let us write

$$\partial_{p,q} = \sum_{r\geqslant 1} \partial^r_{p,q}\colon E^1_{p,q} \to \oplus_{r\geqslant 1} E^r_{p-r,q+r-1}.$$

(This has just finitely many nonzero terms since $\partial^r_{p,q} = 0$ for $r \geqslant p$.) Also define, for $k \in \mathbb{Z}$,

$$D_k = \bigoplus_{p+q=k} E^1_{p,q},$$

and define $\delta_k\colon D_k \to D_{k-1}$ as the map which restricts to $\partial_{p,q}$ on the respective summands $E^1_{p,q}$. Each D_k has a filtration given by

$$F_s D_k = \bigoplus_{p+q=k, p\leqslant s} E^1_{p,q},$$

which is consistent with (A.1) by (A.2). By definition, the map δ_k respects this filtration, and indeed satisfies the stronger property $\delta_k(F_s D_k) \subset F_{s-1}D_{k-1}$.

We will now compute the kernel and image of δ_k. For a general element $x = \sum_p x_p \in D_k$ where each $x_p \in E^1_{p,k-p}$, the component of $\delta_k x$ in the summand $E^1_{m,k-1-m} \subset D_{k-1}$ is equal to

$$\sum_r \partial^r_{m+r,k-m-r} x_{m+r}.$$

Now $\partial^r_{m+r,k-m-r} x_{m+r}$ lies in the subspace $B^{r+1}_{m,k-1-m} \cap H^r_{m,k-1-m}$ of $E^1_{m,k-1-m}$. But these latter subspaces are independent as r varies: indeed given finitely many elements $y_r \in B^{r+1}_{m,k-1-m} \cap H^r_{m,k-1-m}$ that are not all zero, if $r_{\max}$ is chosen maximal subject to the property that $y_{r_{\max}} \neq 0$ then the fact that $0 \neq y_{r_{\max}} \in H^{r_{\max}}_{m,k-1-m}$ while for all $s < r_{\max}$ we have $y_s \in B^{s+1}_{m,k-1-m} \subset B^{r_{\max}}_{m,k-1-m}$ would imply that $\sum y_r \neq 0$ since $H^{r_{\max}}_{m,k-1-m}$ is complementary to $B^{r_{\max}}_{m,k-1-m}$.

The independence of these subspaces implies that, for $x_p \in E^1_{p,k-p}$, the component of $\delta_k\left(\sum_p x_p\right)$ in $E^1_{m,k-m-1}$ is zero only if each $\partial^r_{m+r,k-m-r} x_{m+r}$ separately vanishes. Thus:

$$\sum_p x_p \in \ker \delta_k \Leftrightarrow (\forall p, r)(\partial^r_{p,k-p} x_p = 0). \tag{A.4}$$

Now fixing p and recalling that $Z^p_{p,k-p} = Z^\infty_{p,k-p}$ and $\partial^r_{p,k-p} = 0$ for $r \geqslant p$, note that we have

$$E^1_{p,k-p} = Z^\infty_{p,k-p} \oplus M^{p-1}_{p,k-p} \oplus \cdots \oplus M^1_{p,k-p}.$$

Moreover, for $r < p$, $\partial^r_{p,k-p}$ vanishes on $Z^{r+1}_{p,k-p} \supset Z^\infty_{p,k-p}$ and on each $M^j_{p,k-p}$ for $j \neq r$ while restricting injectively to $M^r_{p,k-p}$. Hence $\partial^r_{p,k-p} x_p = 0$ for *all* r if and only if $x_p \in Z^\infty_{p,k-p}$. In combination with (A.4) this shows:

Proposition A.5.

$$\ker(\delta_k\colon D_k \to D_{k-1}) = \bigoplus_p Z^\infty_{p,k-p}$$

and, for each $s \in \mathbb{N}$,

$$\ker(\delta_k|_{F_s D_k}) = \bigoplus_{p \leqslant s} Z^\infty_{p,k-p}.$$

Next we will show:

Proposition A.6.

$$\operatorname{Im}(\delta_k\colon D_k \to D_{k-1}) = \bigoplus_p B^\infty_{p,k-1-p}$$

and, for $s \in \mathbb{N}$,

$$\operatorname{Im}(\delta_k|_{F_s D_k}) = \bigoplus_{p<s} B^{s-p+1}_{p,k-1-p}.$$

Proof. As noted earlier the summand of $\delta_k\left(\sum_p x_p\right)$ in $E^1_{m,k-1-m}$ is $\sum_r \partial^r_{m+r,k-m-r} x_{m+r}$, which is a sum of terms in the mutually independent subspaces $B^{r+1}_{m,k-1-m} \cap H^r_{m,k-1-m}$. Note that, for fixed k, m and any $t \in \mathbb{N}$,

$$\bigoplus_{1 \leqslant r \leqslant t} \left(B^{r+1}_{m,k-1-m} \cap H^r_{m,k-1-m}\right) = B^{t+1}_{m,k-1-m} : \tag{A.5}$$

indeed using the inclusions

$$B^r_{m,k-1-m} \subset B^{r+1}_{m,k-1-m} \subset Z^r_{m,k-1-m} = H^r_{m,k-1-m} \oplus B^r_{m,k-1-m}$$

we see that $B^{r+1}_{m,k-1-m} = (B^{r+1}_{m,k-1-m} \cap H^r_{m,k-1-m}) \oplus B^r_{m,k-1-m}$; applying this inductively starting from $B^1_{m,k-1-m} = \{0\}$ yields (A.5). The same reasoning shows that $\bigoplus_{r=1}^\infty \left(B^{r+1}_{m,k-1-m} \cap H^r_{m,k-1-m}\right) = B^\infty_{m,k-1-m}$. Thus to prove the proposition it suffices to show that, given $p \in \mathbb{N}$ and elements $y_r \in B^{r+1}_{p-r,k+r-p-1} \cap H^r_{p-r,k+r-p-1}$ for $1 \leqslant r < p$, we can find a single $x \in E^1_{p,k-p}$ with $\partial^r_{p,k-p} x = y_r$ for each r. But this is an easy consequence of Corollary A.4: using the decomposition $E^1_{p,k-p} = Z^\infty_{p,k-p} \oplus M^{p-1}_{p,k-p} \oplus \cdots \oplus M^1_{p,k-p}$ we can take x to be an element with trivial component in $Z^\infty_{p,k-p}$ and with component in each respective $M^r_{p,k-p}$ equal to a preimage of y_r under $\partial^r_{p,k-p}$. □

Corollary A.7. Let $D_* = \oplus_k D_k$ and $\delta = \oplus_k \delta_k$. Then (D_*, δ) is a filtered chain complex whose total homology is given by

$$H_k(D_*, \delta) = \frac{\oplus_p Z^\infty_{p,k-p}}{\oplus_p B^\infty_{p,k-p}}.$$

Moreover, for $s \in \mathbb{N}, t \in \mathbb{N} \cup \{\infty\}$ with $s \leqslant t$ we have

$$\operatorname{Im}\left(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)\right) = \frac{\bigoplus_{p \leqslant s} Z^{\infty}_{p,k-p}}{\bigoplus_{p \leqslant s} B^{t-p+1}_{p,k-p}}.$$

Proof. That (D_*, δ) is a chain complex simply results from Propositions A.5 and A.6 and the fact that $B^{\infty}_{p,q} \subset Z^{\infty}_{p,q}$; the computation of $H_k(D_*, \delta)$ likewise follows immediately. The computation of $\operatorname{Im}\left(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)\right)$ also follows because this image is essentially by definition equal to the quotient of $\ker(\delta|_{F_s D_k})$ by $\operatorname{Im}(\delta|_{F_t D_{k+1}}) \cap F_s D_k$. (For the case that $t = s$, it perhaps also bears noting that $B^1_{s,k-s} = \{0\}$, so that $\oplus_{p<s} B^{s-p+1}_{p,k-p} = \oplus_{p \leqslant s} B^{s-p+1}_{p,k-p}$). □

Lemma A.1 now follows almost immediately from Corollary A.7 and Proposition A.3. Indeed, projecting away $\hat{B}^1_{p,k-p}$ gives isomorphisms $\frac{\hat{Z}^r_{p,k-p}}{\hat{B}^r_{p,k-p}} \cong \frac{Z^r_{p,k-p}}{B^r_{p,k-p}}$ so Corollary A.7 and Proposition A.3 show that we have, whenever $s \in \mathbb{N}$ and $1 \leqslant s \leqslant t \leqslant \infty$,

$$\operatorname{Im}(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)) \cong \bigoplus_{p=1}^{s} \frac{\operatorname{Im}\left(H_k(F_p C_*, \partial) \to H_k(F_t C_*, \partial)\right)}{\operatorname{Im}\left(H_k(F_{p-1} C_*, \partial) \to H_k(F_t C_*, \partial)\right)}.$$

Since $F_0 C_* = \{0\}$, we can then iteratively choose complements to $\operatorname{Im}\left(H_k(F_{p-1} C_*, \partial) \to H_k(F_t C_*, \partial)\right)$ in $\operatorname{Im}\left(H_k(F_p C_*, \partial) \to H_k(F_t C_*, \partial)\right)$ to obtain an isomorphism $\operatorname{Im}\big(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)\big) \cong \operatorname{Im}\left(H_k(F_s C_*, \partial) \to H_k(F_t C_*, \partial)\right)$. Moreover in the case that $t = \infty$, as s varies this can be done in such a way that if $s < s'$ then the isomorphism $\operatorname{Im}\left(H_k(F_{s'} D_*, \delta) \to H_k(D_*, \delta)\right) \cong \operatorname{Im}\left(H_k(F_{s'} C_*, \partial) \to H_k(C_*, \partial)\right)$ restricts to $\operatorname{Im}\left(H_k(F_s D_*, \delta) \to H_k(D_*, \delta)\right)$ as the already-chosen isomorphism $\operatorname{Im}\left(H_k(F_s D_*, \delta) \to H_k(D_*, \delta)\right) \cong \operatorname{Im}\left(H_k(F_s C_*, \partial) \to H_k(C_*, \partial)\right)$; hence by taking the union over s we obtain an isomorphism $H_k(D_*, \delta) \cong H_k(C_*, \delta)$ (corresponding to the case $s = t = \infty$ in Lemma A.1).

Since we have already seen that our complex (D_*, δ) satisfies the other required properties, this completes the proof of Lemma A.1.

A.2. **Construction of $CC_*(X, \lambda)$.** Since we assume that the Reeb flow on the boundary of (X, λ) is nondegenerate, the set of actions (equivalently, periods) of the Reeb orbits on ∂X is discrete; of course every element of this set is positive, so let us denote by $T_1 < T_2 < \cdots < T_r < \cdots$ the numbers which arise as actions of Reeb orbits on ∂X. Also write $T_0 = 0$. By the action filtration property and Proposition 5.5, the maps $\iota_{L_1,L_2} \colon CH^{L_1}(X, \lambda) \to CH^{L_2}(X, \lambda)$ give a directed system (*i.e.* $\iota_{L_2,L_3} \circ \iota_{L_1,L_2} = \iota_{L_1,L_3}$), and ι_{L_1,L_2} is an isomorphism if the interval $(L_1, L_2]$ does not contain any of the actions T_i. So in particular if $L \leqslant L'$ with

$L \in [T_i, T_{i+1})$, $L' \in [T_j, T_{j+1})$ then there is a commutative diagram

$$\begin{array}{ccc} CH^L(X,\lambda) & \xrightarrow{\iota_{L,L'}} & CH^{L'}(X,\lambda) \\ \iota_{T_i,L}\uparrow\cong & & \cong\uparrow\iota_{T_j,L'} \\ CH^{T_i}(X,\lambda) & \xrightarrow[\iota_{T_i,T_j}]{} & CH^{T_j}(X,\lambda) \end{array} \tag{A.6}$$

where both vertical arrows are isomorphisms. So to understand the maps ι_{L_1,L_2} it suffices to understand the maps ι_{T_i,T_j}.

By Definition 5.1, we have

$$CH^L(X,\lambda) = \varinjlim_{N,H} HF^{S^1,N,+,\leqslant L}(H,J)$$

where the direct limit is taken over parametrized Hamiltonians $H\colon S^1 \times \hat{X} \times S^{2N+1} \to \mathbb{R}$ on the Liouville completion $\hat{X}$ of X that satisfy a certain admissibility condition, with the structure maps being given by parametrized versions of continuation maps associated to pairs $(N,H),(N'H')$ with $N \leqslant N', H \leqslant H'|_{S^1\times\hat{X}\times S^{2N+1}}$. Here $HF^{S^1,N,+,\leqslant L}(H,J)$ is the homology of the subcomplex (which for brevity we will denote by $C(N,H)^L$) generated by orbits of symplectic action at most L of the positive equivariant Floer complex $C(N,H)^\infty := \frac{CF^{S^1,N,+}(N,H)}{CF^{S^1,N,+,\leqslant\varepsilon}(N,H)}$ where $0 < \varepsilon \ll T_1$. The maps $\iota_{L_1,L_2}\colon CH^{L_1}(X,\lambda) \to CH^{L_2}(X,\lambda)$ are by definition the maps induced on the direct limit by the maps $HF^{S^1,N,+,\leqslant L_1}(N,H) \to HF^{S^1,N,+,\leqslant L_2}(N,H)$ given by the inclusion of subcomplexes $C(N,H)^{L_1} \hookrightarrow C(N,H)^{L_2}$.

Suppose that $\{(N_i,H_i)\}_{i=1}^\infty$ is any cofinal, linearly ordered subset of the partially ordered set of pairs (N,H) used to define $CH^L(X,\lambda)$. We can then form the direct limit of the chain complexes $C(N_i,H_i)^\infty$, using as structure maps the compositions of chain level continuation maps $C(N_i,H_i)^\infty \to C(N_{i+1},H_{i+1})^\infty$. Denote this direct limit by $\underrightarrow{C}$. Since the continuation maps preserve the filtration by symplectic action, for any $L \in \mathbb{R}$ we likewise have a direct limit $\underrightarrow{C}^L = \varinjlim_i C(N_i,H_i)^L$, and the $\underrightarrow{C}^L$ form an $\mathbb{R}$-valued filtration of $\underrightarrow{C}$.

Let us coarsen this $\mathbb{R}$-filtration to an $\mathbb{N}$ filtration by, for each $p \in \mathbb{N}$, choosing T'_p with $T_p < T'_p < T_{p+1}$, and letting

$$F_p\underrightarrow{C} = \underrightarrow{C}^{T'_p}$$

(Recall our notation that $T_0 = 0$ and the T_p for $p > 0$ are the distinct actions of Reeb orbits along ∂X, in increasing order.) As in Remark 2.8, for i sufficiently large every generator of $C(N_i,H_i)$ will have filtration level greater than T'_0, so that $C(N_i,H_i)^{T'_0} = \{0\}$ for i sufficiently large and so $F_0\underrightarrow{C} = \{0\}$. The fact that $\cup_p C(N_i,H_i)^{T'_p} = C(N_i,H_i)$ for each i implies that likewise $\cup_p F_p\underrightarrow{C} = \underrightarrow{C}$. All of our complexes are $\mathbb{Z}$-graded because of the assumption that $c_1(TX)|_{\pi_2(X)} = 0$. Thus Lemma A.1 applies to the filtered complex $\underrightarrow{C}$, producing a filtered complex (D_*, δ) with

$$F_r D_* = \bigoplus_{1 \leqslant p \leqslant r} H_* \left(\frac{\varinjlim_i C(N_i, H_i)^{T'_p}}{\varinjlim_i C(N_i, H_i)^{T'_{p-1}}} \right)$$

such that for each $k \in \mathbb{Z}, s \leqslant t$ we have

$$\operatorname{Im}(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)) \cong \operatorname{Im}\left(H_k(\underrightarrow{C}^s) \to H_k(\underrightarrow{C}^t) \right);$$

note that (for finite t) the right-hand side is precisely the image of ι_{T_s, T_t} in grading k. Also, since $\varinjlim$ is an exact functor, we have

$$H_* \left(\frac{\varinjlim_i C(N_i, H_i)^{T'_p}}{\varinjlim_i C(N_i, H_i)^{T'_{p-1}}} \right) \cong \varinjlim_i H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right).$$

Thus we have a filtered complex (D_*, δ) whose r-filtered part is

$$F_r D_* = \varinjlim_i \bigoplus_{1 \leqslant p \leqslant r} H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right)$$

and such that, for $1 \leqslant s \leqslant t < \infty$,

(A.7)

$$\operatorname{Im}\left(\iota_{T'_s, T'_t} \colon CH_k^{T'_s}(X, \lambda) \to CH_k^{T'_t}(X, \lambda) \right) \cong \operatorname{Im}(H_k(F_s D_*, \delta) \to H_k(F_t D_*, \delta)).$$

The foregoing discussion applies to an arbitrary cofinal linearly ordered subset $\{(N_i, H_i)\}_{i=1}^{\infty}$ of the set of admissible pairs (N, H). For a particular choice of such a cofinal subset consisting of Hamiltonians as described in [Gut17, Section 3.1] and [GH17, Remark 5.15], the homologies $H_* \left(\frac{C(N_i,H_i)^{T'_p}}{C(N_i,H_i)^{T'_{p-1}}} \right)$ are computed in [Gut17, Section 3.2],[GH17, Section 6.7]. Namely, the space $H_* \left(\frac{C(N_i,H_i)^{T'_p}}{C(N_i,H_i)^{T'_{p-1}}} \right)$ is generated by elements $\check{\gamma}$ and $u^{N_i} \otimes \hat{\gamma}$ as γ ranges over Reeb orbits having action equal to T_p; writing CZ for the Conley–Zehnder index, the grading of $\check{\gamma}$ is $CZ(\gamma)$ and that of $u^{N_i} \otimes \hat{\gamma}$ is $CZ(\gamma) + 2N_i + 1$. The continuation maps $H_* \left(\frac{C(N_i,H_i)^{T_p}}{C(N_i,H_i)^{T_{p-1}}} \right) \to H_* \left(\frac{C(N_{i+1},H_{i+1})^{T_p}}{C(N_{i+1},H_{i+1})^{T_{p-1}}} \right)$ moreover map $\check{\gamma}$ to $\check{\gamma}$ and $u^{N_i} \otimes \hat{\gamma}$ to zero. Thus in any given degree k the direct limit $\varinjlim_i H_k \left(\frac{C(N_i,H_i)^{T'_p}}{C(N_i,H_i)^{T'_{p-1}}} \right)$ has basis in bijection with the Reeb orbits on ∂X of action T_p and Conley-Zehnder index k.

So the $\mathbb{N}$-filtered complex (D_*, δ) produced by Lemma A.1 has the property that $F_r D_k$ is the span of a set of generators in bijection with the Reeb orbits on ∂X of Conley–Zehnder index k and action at most T_r. The complex $CC_*(X, \lambda)$ promised in the Reeb orbit property is then given by converting (D_*, δ) into an $\mathbb{R}$-filtered complex by taking the L-filtered part $CC_*^L(X, \lambda)$ to be equal $F_r D_*$ where r is maximal subject to the condition that $T_r \leqslant L$. In particular we have equalities $CC_*^L(X,\lambda) = CC_*^{L'}(X,\lambda)$ whenever $L, L' \in [T_r, T_{r+1})$. Since δ strictly decreases the $\mathbb{N}$-filtration on D_*, it likewise strictly decreases this $\mathbb{R}$-filtration.

By (A.7), we have isomorphisms

$$\operatorname{Im}\left(\iota_{T'_s,T'_t}\colon CH_k^{T'_s}(X,\lambda) \to CH_k^{T'_t}(X,\lambda)\right) \cong \operatorname{Im}\left(H_k(CC_*^{T'_s}(X,\lambda)) \to H_k(CC_*^{T'_t}(X,\lambda))\right)$$

for $s \leqslant t$, and then by applying (A.6) we obtain a similar isomorphism with T'_s, T'_t replaced by arbitrary L, L' with $L \leqslant L'$. The special case that $L = L'$ shows that $CH_k^L(X,\lambda)$ is isomorphic to $H_k(CC_*(X,\lambda))$. This completes the proof that the filtered complex $CC_*(X,\lambda) = \cup_L CC_*^L(X,\lambda)$ with boundary operator δ satisfies the properties required by the Reeb orbit property.

Appendix B. Technical Lemmas

B.1. My favorite maximum principle.

Theorem B.1 (Abouzaid, [Rit13]). Let $(W', \omega' = d\lambda')$ be an exact symplectic manifold with contact type boundary $\partial W'$, such that the Liouville vector field points inwards. Let ρ be the coordinate near $\partial W'$ defined by the flow of the Liouville vector field starting from the boundary and let $r := e^\rho$; near the boundary the symplectic form writes $\omega' = d(r\alpha)$ with α the contact form on $\partial W'$ given by the restriction of λ'. Let J be a compatible almost complex structure such that $J^*\lambda' = dr$ on the boundary.
a) Let $H : W' \to \mathbb{R}$ be non negative, and such that $H = h(r)$ where h is a convex increasing function near the boundary. Let S be a compact Riemann surface with boundary and let β be a 1-form such that $d\beta \geqslant 0$. Then any solution $u : S \to W'$ of $(du - X_H \otimes \beta)^{0,1} = 0$ with $u(\partial S) \subset \partial W'$ is entirely contained in $\partial W'$.
b) Let $H : \mathbb{R} \times S^1 \times W' \to \mathbb{R}$ be an increasing homotopy, such that $H(s,\theta,p,\rho) = H_s^\theta(p,\rho) = h_s(r)$ where h_s are convex increasing functions near the boundary. Let S be a compact Riemann surface with boundary embedded in the cylinder ($\mathbb{R} \times S^1$ with the standard structure). Then any solution $u : S \to W'$ of $(du - X_{H_s} \otimes d\theta)^{0,1} = 0$ with $u(\partial S) \subset \partial W'$ is entirely contained in $\partial W'$.

Proof. Proof of part a. The energy of a map $u : S \to W'$ is defined as $E(u) := \frac{1}{2}\int_S \|du - X_H \otimes \beta\|^2 vol_S$ where du is viewed as a section of $T^\star S \otimes u^\star TW'$. If $s + it$ is a local holomorphic coordinate on S, so that $j(\partial_s) = \partial_t$ and $vol_S = ds \wedge dt$ we have

$$\begin{aligned}\frac{1}{2}\|du - X_H \otimes \beta\|^2 vol_S &= \omega'\left(\partial_s u - X_H\beta(\partial_s), \partial_t u - X_H\beta(\partial_t)\right) ds \wedge dt \\ &= \left(\omega'(\partial_s u, \partial_t u) - dH(\partial_t u)\beta(\partial_s) + dH(\partial_s u)\beta(\partial_t)\right) ds \wedge dt \\ &= u^\star\omega' + u^\star(dH) \wedge \beta.\end{aligned}$$

It is obviously non negative for any path. Since $d(u^\star H\beta) = u^\star(dH)\wedge\beta + \underbrace{u^\star H d\beta}_{\geqslant 0}$, we have

$$
\begin{aligned}
E(u) = \int_S u^\star d\lambda' + u^\star(dH)\wedge\beta \leqslant \int_S d(u^\star\lambda') + d(u^\star H\beta) \leqslant \int_{\partial S} u^\star\lambda' - \lambda'(X_H)\beta \\
\text{since } H = h(r) \leqslant rh'(r) = r\alpha\big(h'(r)R_\alpha\big) = -\lambda'(X_H) \text{ on } u(\partial S)\subset\partial V \\
= \int_{\partial S}\lambda'(du - X_H\otimes\beta) \\
= \int_{\partial S} -\lambda' J(du - X_H\otimes\beta)j \qquad \text{since } (du - X_H\otimes\beta)^{0,1} = 0 \\
= \int_{\partial S} -dr(du - X_H\otimes\beta)j \quad \text{since } J^\star\lambda' = dr \text{ along } u(\partial S)\subset\partial W' \\
= \int_{\partial S} -dr\, du\, j \qquad \text{since } dr \text{ vanishes on } X_H \text{ on } u(\partial S)\subset\partial W'.
\end{aligned}
$$

Let ν be the outward normal direction along ∂S. Then $(\nu, j\nu)$ is an oriented frame, so ∂S is oriented by $j\nu$. Now $dr(du)j(j\nu) = d(r\circ u)(-\nu) \geqslant 0$ since in the inward direction, $-\nu$, $r\circ u$ can only increase because r is minimum on $\partial W'$. So $E(u)\leqslant 0$ hence $E(u) = 0$. This implies that $du - X_H\otimes\beta = 0$ which shows that the image of du is in the span of X_H which is the span of $R_\alpha \in T\partial W'$ on $\partial W'$. Hence the image of u is entirely in contained in $\partial W'$.
Proof of part b. The proof starts as above. The energy of u is non negative and given by

$$E(u) := \frac{1}{2}\int_S \|du - X_{H_s}\otimes d\theta\|^2 vol_S = \int_S u^\star\omega' + u^\star(dH_s^\theta)\wedge d\theta.$$

We have $u^\star(dH_s^\theta)\wedge d\theta = d(u'^\star H)\wedge d\theta - \underbrace{u^\star\partial_s H_s^\theta ds\wedge d\theta}_{\geqslant 0}$, for $u' : S\to\mathbb{R}\times S^1\times W'$ which maps an element $(\theta, s)\in S$ to the element $(s,\theta,u'(\theta,s))$. Hence

$$
\begin{aligned}
E(u) &= \int_S u^\star d\lambda' + u^\star(dH)\wedge d\theta \\
&\leqslant \int_S d(u^\star\lambda') + d(u'^\star H d\theta) \\
&\leqslant \int_{\partial S} u^\star\lambda' - \lambda'(X_{H_s})d\theta \text{ using Stokes's theorem and} \\
&\qquad H = h_s(r) \leqslant r\alpha\big(h_s'(r)R_\alpha\big) = -\lambda'(X_{H_s}) \text{ on } u(\partial S)\subset\partial V \\
&= \int_{\partial S}\lambda'(du - X_{H_s}\otimes d\theta)
\end{aligned}
$$

and the proof proceeds as in part a. □

B.2. **Parenthesis on the homotpy of homotopies Theorem.** This classical material can be found, for instance, in [AD14, Sal99, Rit].

Definition B.2. Let H_1 and H_2 be two Hamiltonians. We say that an increasing homotopy H_s between H_1 and H_2 is *regular* if for all 1-periodic orbits $\gamma_1 \in \mathscr{P}(H_1)$ and $\gamma_2 \in \mathscr{P}(H_2)$, $\mathscr{M}(\gamma_1, \gamma_2, H_s, J_s)$ is a manifold of dimension $-\mathrm{CZ}(\gamma_1) + \mathrm{CZ}(\gamma_2)$.

Theorem B.3. The morphism

$$\phi : SH(H_1, J_1) \to SH(H_2, J_2)$$

is independent of the choice of the regular homotopy between H_1 and H_2.

Proof. Consider two regular homotopies K_0 and K_1 joining H_1 and H_2. We are going to construct an homotopy between ϕ^{K_0} and ϕ^{K_1} in other word a

$$S : SC_*(H_1, J_1) \to SC_{*+1}(H_2, J_2)$$

satisfying the relation

$$\phi^{K_1} - \phi^{K_0} = S \circ \partial_{H_1} + \partial_{H_2} \circ S.$$

Consider a homotopy of homotopies K_η, $\eta \in [0,1]$ such that in a neighbourhood of 0, $K_\eta \equiv K_0$ and in a neighbourhood of 1, $K_\eta \equiv K_1$. For $\gamma_1 \in \mathscr{P}(H_1)$, $\gamma_2 \in \mathscr{P}(H_2)$ and η fixed, we denote by $\mathscr{M}(\gamma_1, \gamma_2, K_\eta)$ the space of Floer trajectories $u : \mathbb{R} \times S^1 \to \mathbb{R}$

$$\partial_s u + J_\eta(\partial_\theta u - X_{K_\eta}) = 0$$

and define the parametrized moduli space

$$\mathscr{M}^K(\gamma_1, \gamma_2) := \bigcup_{\eta \in [0,1]} \mathscr{M}(\gamma_1, \gamma_2, K_\eta).$$

We now use the following two theorems:

Theorem B.4. If $\mathrm{CZ}(\gamma_1) - \mathrm{CZ}(\gamma_2) + 1 = 0$, then $\mathscr{M}^K(\gamma_1, \gamma_2)$ is a compact manifold of dimension 0.

Theorem B.5 ([AD14]). Let us define

$$\Pi^K(\gamma_1, \gamma_2) := \left(\bigcup_{\substack{\gamma_1' \in \mathscr{P}(H_1) \\ \mathrm{CZ}(\gamma_1) - \mathrm{CZ}(\gamma_1') = 1}} \mathscr{M}(\gamma_1, \gamma_1', H_1, J_1)/_{\mathbb{R}} \times \mathscr{M}^K(\gamma_1, \gamma_1') \right)$$
$$\bigcup \left(\bigcup_{\substack{\gamma_2' \in \mathscr{P}(H_2) \\ \mathrm{CZ}(\gamma_2') - \mathrm{CZ}(\gamma_2) = 1}} \mathscr{M}^K(\gamma_2', \gamma_2') \times \mathscr{M}(\gamma_2', \gamma_2, H_2, J_2)/_{\mathbb{R}} \right).$$

If $\mathrm{CZ}(\gamma_1) = \mathrm{CZ}(\gamma_2)$, then $\mathscr{M}^K(\gamma_1,\gamma_2) \cup \Pi^K(\gamma_1,\gamma_2)$ is a compact manifold of dimension 1 with boundary equal to

$$\Pi^K(\gamma_1,\gamma_2) \cup \underbrace{\big(\{0\}\times\mathscr{M}(\gamma_1,\gamma_2,K_0)\big)}_{\text{with opposite orientation}} \cup \big(\{1\}\times\mathscr{M}(\gamma_1,\gamma_2,K_1)\big).$$

We are now ready to proceed with the proof of Theorem B.3. We define the homotopy $S : SC_*(H_1,J_1) \to SC_{*+1}(H_2,J_2)$ as follows: if $\gamma_1 \in \mathscr{P}(H_1)$ such that $\mathrm{CZ}(\gamma_1) = k$ then

$$S_k(\gamma_1) = \sum_{\substack{\gamma_2\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2)=k+1}} \#\mathscr{M}^K(\gamma_1,\gamma_2)\gamma_2.$$

We have, for $\gamma_1 \in \mathscr{P}(H_1)$ such that $\mu_{CZ}(\gamma_1) = k$,

$$\begin{aligned}
S\circ\partial_{H_1}(\gamma_1) &+ \partial_{H_2}\circ S(\gamma_1)\\
&= S_{k-1}\sum_{\substack{\gamma_1'\in\mathscr{P}(H_1)\\ \mathrm{CZ}(\gamma_1')=k-1}} \big(\#\mathscr{M}(\gamma_1,\gamma_1',H_1,J_1)/_{\mathbb{R}}\big)\gamma_1' + \partial_{H_2}\sum_{\substack{\gamma_2'\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2')=k+1}} \#\mathscr{M}^K(\gamma_1,\gamma_2')\gamma_2'\\
&= \sum_{\substack{\gamma_2\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2)=k}}\ \sum_{\substack{\gamma_1'\in\mathscr{P}(H_1)\\ \mathrm{CZ}(\gamma_1')=k-1}} \big(\#\mathscr{M}(\gamma_1,\gamma_1',H_1,J_1)/_{\mathbb{R}}\big)\#\mathscr{M}^K(\gamma_1,\gamma_2)\gamma_2\\
&\qquad + \sum_{\substack{\gamma_2\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2)=k}}\ \sum_{\substack{\gamma_2'\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2')=k+1}} \#\mathscr{M}^K(\gamma_1,\gamma_2')\big(\#\mathscr{M}(\gamma_2',\gamma_2,H_2,J_2)/_{\mathbb{R}}\big)\gamma_2\\
&= \sum_{\substack{\gamma_2\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2)=k}} \#\Pi^K(\gamma_1,\gamma_2)\gamma_2.
\end{aligned}$$

On the other side

$$\phi^{K_1} - \phi^{K_0} = -\sum_{\substack{\gamma_2\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2)=k}} \big(\#\mathscr{M}(\gamma_1,\gamma_2,K_0)\big)\gamma_2 + \sum_{\substack{\gamma_2\in\mathscr{P}(H_2)\\ \mathrm{CZ}(\gamma_2)=k}} \big(\#\mathscr{M}(\gamma_1,\gamma_2,K_1)\big)\gamma_2.$$

Therefore we reach the conclusion using theorem B.5. □

B.3. **Parenthesis : A glimpse on signs.** We indicate in this section how to define the signs attached to Floer trajectories, and mention the steps which prove that $\#\partial\mathscr{M}/_{\mathbb{R}} = 0$ (and hence that $\partial^2 = 0$).

B.3.1. *Operator gluing lemma.* We look at operators

$$D : W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n}) \to L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n}),$$

with $p > 2$, which are of the form

$$D = \partial_s + J\partial_t + S(s,t)$$

with $S(s,\cdot) \to S_\pm(\cdot)$ for $s \to \pm\infty$, where $S_\pm$ belongs to the following space $\mathscr{S}$ of loops of symmetric matrices. Given a loop S of symmetric matrices, one considers the corresponding path ψ of symplectic matrices defined by the differential

equations $\dot{\psi} = JS\psi$. The loop S belongs to the space $\mathscr{S}$ if and only if 1 is not an eigenvalue of $\psi(1)$, i.e. $\det\big(\psi(1) - \mathrm{Id}\big) \neq 0$.
We denote by $\mathscr{O}(\mathbb{R} \times S^1; S_-, S_+)$ the space of such operators.

For $R >> 0$ big enough and $S_\pm, S \in \mathscr{S}$, one defines a *gluing operation*

$$\mathscr{O}(\mathbb{R} \times S^1; S_-, S) \times \mathscr{O}(\mathbb{R} \times S^1; S, S_+) \to \mathscr{O}(\mathbb{R} \times S^1; S_-, S_+) : (D_1, D_2) \mapsto D_1 \#_R D_2$$

in the following way. Fix a smooth function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(s) = 0$ for $s \leqslant 0$, $\beta(s) = 1$ for $s \geqslant 1$.Define,

$$D_i^R := \partial_s + J\partial_t + S(t) + \beta(-s+R)\big(S_i(s,t) - S(t)\big) \quad \text{for } i = 1,2$$

The glued operator, $D_1 \#_R D_2$ is defined by

$$D_1 \#_R D_2 := \begin{cases} D_1^R(s+R) & \text{if } s \leqslant 0 \\ D_2^R(s-R) & \text{if } s \geqslant 0 \end{cases}.$$

Theorem B.6 (Operator gluing lemma)**.** Assume D_1 and D_2 are surjective. Then $D_1 \#_R D_2$ is surjective for $R >> 0$ and has uniformly bounded right inverse.

Proof. Choose Q_1, Q_2, right inverse for D_1, D_2. We first construct an approximate right inverse T_R for $D_R = D_1 \#_R D_2$ i.e an operator

$$T_R : L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

such that

$$\|D_R T_R - \mathrm{Id}\| < \tfrac{1}{2} \quad \text{and} \quad T_R \text{ is uniformly bounded.}$$

Then $Q_R := T_R(D_R T_R)^{-1}$ is a genuine right inverse for D_R and is uniformly bounded.

We construct T_R according to the following diagram:

$$\begin{array}{ccc} L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) & \xrightarrow{Q_1 \times Q_2} & W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \\ \uparrow S_R & & \downarrow G_R \\ L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) & \overset{T_R}{\dashrightarrow} & W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \end{array}$$

We define $T_R := G_R \circ Q_1 \times Q_2 \circ S_R$ where G_R is a "gluing map" and S_R is a "splitting map". The splitting map is defined as

$$S_R(\zeta) = (\zeta_1, \zeta_2) \quad \text{with} \begin{cases} \zeta_1(s,\cdot) = \big(1 - \beta(s-R)\big)\zeta(s-R,\cdot) \\ \zeta_2(s,\cdot) = \beta(s+R)\zeta(s+R,\cdot) \end{cases}.$$

Given $L > 0$, we define $\beta_L(s) := \beta\left(\frac{s}{L}\right)$ and we assume that β is such that $\beta_L(s) = 0$ for $s \in [0,1]$ if L is big. We define the gluing map to be

$$G_R(\xi_1, \xi_2) = \Big(1 - \beta_{\frac{R}{2}}(s)\Big)\xi_1(s+R) + \Big(1 - \beta_{\frac{R}{2}}(-s)\Big)\xi_2(s-R).$$

Note that since G_R and S_R are uniformly bounded, so is T_R.
To conclude the theorem, one can show that

$$\|D_R T_R - \mathrm{Id}\| \to 0 \text{ as } R \to \infty.$$

□

B.3.2. *Coherent orientations.* Let $D : X \to Y$ be a Fredholm operator. Its determinant is the 1-dimensional vector space

$$\det(D) := \Lambda^{\max} \ker(D) \otimes \Lambda^{\max} \big(\operatorname{coker}(D)\big)^{\vee}.$$

An orientation of D is an orientation of this vector space.
One considers the real line bundle $\det \to \mathscr{F}(X, Y)$ over the space of Fredholm operators, whose fiber above D is $\det(D)$.

Lemma B.7. Assume $D_1 \in \mathscr{O}(\mathbb{R} \times S^1; S_-, S)$ and $D_2 \in \mathscr{O}(\mathbb{R} \times S^1; S, S_+)$ are surjective. For R big enough, if $D_R := D_1 \#_R D_2$ and Q_R is its right inverse, there is a canonical isomorphism

$$\phi_R : \ker(D_1) \oplus \ker(D_2) \to \ker(D_R)$$

defined by

$$\phi_R := (\mathrm{Id} - Q_R D_R) \circ G_R$$

where G_R is the gluing map defined above.

A consequence of this Lemma is that there is a canonical isomorphism

$$\det(D_1) \otimes \det(D_2) \to \det(D_R) \quad \text{for } R \gg 0. \tag{B.1}$$

Definition B.8. A *system of coherent orientations on the space of operators*

$$\{\mathscr{O}(\mathbb{R} \times S^1; S_-, S_+) | S_-, S_+ \text{ as above } \}$$

is an orientation of the determinant line bundle over each $\mathscr{O}(\mathbb{R} \times S^1; S_-, S_+)$, which is compatible with the gluing operation via the canonical isomorphism (B.1):

$$\det(D_1 \#_R D_2) \simeq \det(D_1) \otimes \det(D_2).$$

This can be done because $\mathscr{O}(\mathbb{R} \times S^1; S_-, S_+)$ is contractible.

Theorem B.9 ([FH94, BM04, BO09])**.** There exists a system of coherent orientations.

Definition B.10. Assume a system of coherent orientations is given. We shall define the sign attached in Floer coboundary operator to a Floer trajectory between two 1- periodic orbits with a difference of Conley-Zehnder index equal to 1. The space of trajectories is of dimension 1, its quotient by the action of $\mathbb{R}$, $\mathscr{M}(\gamma^-, \gamma^+)$, is of dimension 0. Given $[u] \in \mathscr{M}(\gamma^-, \gamma^+)$, the dimension of $\ker(D_u)$ is equal to 1; this $\ker(D_u)$ is spanned by $\langle \partial_s u \rangle$. The *sign associated to* $[u]$, $\varepsilon\big([u]\big)$, is given by

$$\begin{cases} +1 & \text{if orientation of } \ker(D_u) \text{ given by } \langle \partial_s u \rangle \text{ coincides with the coherent orientation} \\ -1 & \text{if it is not the case.} \end{cases}$$

Proposition B.11. Consider two broken Floer trajectories $([u], [v])$ and $([u'], [v'])$ which are the two ends of a 1-dimensional moduli space $\mathscr{M}(x, y)$. Then

$$\varepsilon\big([u]\big) \cdot \varepsilon\big([v]\big) + \varepsilon\big([u']\big) \cdot \varepsilon\big([v']\big) = 0.$$

This shows that $\# \partial \mathscr{M}(x, y) = 0$.

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